

On dual description of η -deformed integrable sigma models

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- Studying QFT's in the strong coupling regime is an interesting problem
- Analysis simplifies if the theory shares the property of the so called weak/strong coupling duality
- Among known examples: Coleman-Mandelstam duality, electric magnetic duality, various string dualities, AdS/CFT duality etc
- Today, we will study another example of this phenomenon. Pair of dual theories is: the so called η -deformed symmetric space sigma model and certain Toda theory.
- Both theories are two-dimensional integrable QFT's.

- We start with an example: $O(3)$ sigma-model

$$S = \frac{1}{g^2} \int \left((\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2 + (\partial_\mu \varphi_3)^2 \right) d^2x, \quad \sum_{k=1}^3 \varphi_k^2 = 1.$$

- This theory is conformal and integrable at the classical level (Pohlmeyer).
- As QFT it corresponds to an asymptotically free theory with a dynamically generated mass scale and is also integrable (Polyakov).
- Its S -matrix is constrained by integrability (Zamolodchikov $\times 2$)

$$S_{ij}^{kl}(\theta) = \delta_{ij}\delta_{kl}S_1(\theta) + \delta_{ik}\delta_{jl}S_2(\theta) + \delta_{il}\delta_{jk}S_3(\theta),$$

$$S_1(\theta) = \frac{2i\pi\theta}{(\theta + i\pi)(\theta - 2i\pi)}, \quad S_2(\theta) = \frac{\theta(\theta - i\pi)}{(\theta + i\pi)(\theta - 2i\pi)}, \quad S_3(\theta) = \dots$$

where $\theta = \theta_1 - \theta_2$ is the rapidity difference of the incoming states.

- This rational S -matrix corresponds to the Yangian $Y(SO(3))$. Yangians always admit one-parametric deformation called the quantum affine group.

- Corresponding S -matrix is a trigonometric one

$$S_{++}^{++}(\theta) = \frac{\sinh \lambda(\theta - i\pi)}{\sinh \lambda(\theta + i\pi)}, \quad S_{+0}^{+0} = \frac{\sinh \lambda\theta}{\sinh \lambda(\theta - 2i\pi)} S_{++}^{++}(\theta), \dots$$

where λ is the deformation parameter. At $\lambda \rightarrow 0$ we are back to the rational case.

- It is naturally to expect that there should exist an integrable deformation of the $O(3)$ SM, which corresponds to this trigonometric S -matrix.
- At least this deformed sigma-model should correspond to renormalizable QFT.

- Analysis of general sigma-model

$$\mathcal{A} = \frac{1}{4\pi} \int \left(G_{\mu\nu}(\varphi) \partial\varphi^\mu \bar{\partial}\varphi^\nu + \dots \right) d^2x,$$

requires the target space metric $G_{\mu\nu}$ to satisfy the Ricci flow equations

$$R_{\mu\nu} + \dots = -\dot{G}_{\mu\nu}, \quad \text{where} \quad \dot{\cdot} = \frac{d}{dt}, \quad t \sim \log \frac{\Lambda^*}{\Lambda}$$

- There is remarkable solution in 2d (Fateev, Onofri, Zamolodchikov)

$$ds^2 = \frac{\kappa}{\nu} \left(\frac{d\zeta^2}{(1-\zeta^2)(1-\kappa^2\zeta^2)} + \frac{(1-\zeta^2)d\phi^2}{(1-\kappa^2\zeta^2)} \right), \quad \kappa = -\tanh \nu t$$

- While embedded in 3d it looks like a “sausage” of length $L = -\sqrt{\nu}t$: infinitely long in the UV ($t \rightarrow -\infty$) and shrinking at the intermediate scale ($t \sim 0$). At $\nu \rightarrow 0$ we are back to the round sphere.
- It has been conjectured that the full theory promoted by the one-loop action is integrable with the trigonometric S -matrix.

- We note that

$$S_{++}^{++}(\theta) = \frac{\sinh \lambda(\theta - i\pi)}{\sinh \lambda(\theta + i\pi)}, \quad S_{+0}^{+0} = \frac{\sinh \lambda\theta}{\sinh \lambda(\theta - 2i\pi)} S_{++}^{++}(\theta), \dots$$

simplifies in the limit $\lambda \rightarrow \frac{1}{2}$: $S_{++}^{++}(\theta) \rightarrow -1 \dots$

- It looks like in the limit we have a theory of free boson and Dirac fermion of the same mass.
- Away from $\lambda = \frac{1}{2}$ one expects some local QFT. Remarkably, it has been guessed by Alyosha Zamolodchikov

$$\mathcal{L} = \frac{1}{8\pi} (\partial_\mu \Phi)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi + \frac{\pi b^2}{2(1+b^2)} (\bar{\psi}\gamma^\mu \psi)^2 - m\bar{\psi}\psi \cosh(b\Phi) - \frac{m^2}{8\pi b^2} \sinh^2(b\Phi),$$

provided that $\lambda = \frac{1}{2(1+b^2)}$.

- As it becomes clear recently the sausage sigma-model belongs to the more general class of the so called η -deformed sigma models.
- It's all started with the seminal Klimciks observation (Klimcik 2008). First, one deforms the PCF model

$$\mathcal{S} = \frac{1}{2} \int \text{Tr} \left(\mathbf{g}^{-1} \partial_+ \mathbf{g} \mathbf{g}^{-1} \partial_- \mathbf{g} \right) d^2 x.$$

This theory has a global $G_L \times G_R$ symmetry which acts as

$$\mathbf{g} \rightarrow U \mathbf{g} V, \quad U, V \in G.$$

This theory is known to be classically/quantum integrable (Zakharov-Mikhailov, Polyakov-Wiegmann).

- Now, let H be the Lie subgroup of G and $\mathfrak{h} = \text{Lie}(H)$. We can obtain the sigma-model on a quotient space G/H by gauging

$$\partial_{\pm} \rightarrow D_{\pm} = \partial_{\pm} - A_{\pm}, \quad A_{\pm} \in \mathfrak{h}.$$

The non-trivial fact is that it preserves integrability (classically!).

- The deformation is given in terms of the linear operator $\mathcal{R}: \mathfrak{g} \rightarrow \mathfrak{g}$

$$\langle a, \mathcal{R}b \rangle = -\langle \mathcal{R}a, b \rangle,$$

and satisfies

$$[\mathcal{R}a, \mathcal{R}b] + \mathcal{R}([a, \mathcal{R}b] + [\mathcal{R}a, b]) - [a, b] = 0,$$

called the modified Yang-Baxter equation.

- With the operator \mathcal{R} at hand we define the deformed PCF action as

$$\mathcal{S} = \frac{1}{2} \int \text{Tr} \left(\mathfrak{g}^{-1} \partial_+ \mathfrak{g} \frac{1}{1 - \eta \mathcal{R}} \mathfrak{g}^{-1} \partial_- \mathfrak{g} \right) d^2x,$$

where η is the deformation parameter ($\eta = i\kappa$). It has been shown by Klimcik that this particular type of deformations survives the integrability.

- Note that the deformed theory is still left G invariant. So, one can apply the coset construction.

- The action of the η -deformed coset sigma model can be written in the form (Delduc et al)

$$\mathcal{S} = \frac{1}{2} \int \text{Tr} \left(\left(\mathbf{g} \partial_+ \mathbf{g}^{-1} \right)^{(c)} \frac{1}{1 - \eta \mathcal{R}_{\mathbf{g}} \circ \mathcal{P}_c} \left(\mathbf{g} \partial_- \mathbf{g}^{-1} \right)^{(c)} \right) d^2 x,$$

where $\mathcal{R}_{\mathbf{g}} = \text{Ad } \mathbf{g}^{-1} \circ \mathcal{R} \circ \text{Ad } \mathbf{g}$ and \mathcal{P}_c is the projection on the coset space.

- It has been shown that this theory is integrable provided that G/H is a symmetric space.
- For example $G/H = SO(N)/SO(N - 1)$ provides an integrable deformation of the $O(N)$ sigma-model.
- As expected $SO(3)/SO(2)$ case corresponds to the deformed $O(3)$ sigma-model and coincides with the sausage model for certain choice of the operator \mathcal{R} .

- One way to explain the duality requires bosonization. Namely, we rewrite

$$\mathcal{L} = \frac{1}{8\pi}(\partial_\mu\Phi)^2 + i\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{\pi b^2}{2(1+b^2)}(\bar{\psi}\gamma^\mu\psi)^2 - m\bar{\psi}\psi \cosh(b\Phi) + \dots$$

in terms of two bosonic fields (here $\beta = \sqrt{1+b^2}$)

$$\mathcal{L} = \frac{1}{8\pi}(\partial_\mu\varphi)^2 + \frac{1}{8\pi}(\partial_\mu\Phi)^2 - m \cos(\beta\varphi) \cosh(b\Phi) + \dots$$

- This theory is integrable $[\mathbf{I}_{2k-1}, \mathbf{I}_{2l-1}] = 0$. Perturbatively

$$\mathbf{I}_{2k-1} = \mathbf{I}_{2k-1}^{\text{free}} + O(m), \quad \text{with} \quad \mathbf{I}_{2k-1}^{\text{free}} = \frac{1}{2\pi} \int_{\mathcal{C}} W_{2k}(\partial\varphi, \partial\Phi) dz,$$

- For example,

$$W_2(\partial\varphi, \partial\Phi) = (\partial\varphi)^2 + (\partial\Phi)^2.$$

- Higher-spin densities can be defined from the requirement that they commute with the perturbation

$$[\mathbf{I}_{2k-1}^{\text{free}}, \int e^{\pm i\beta\varphi \pm b\Phi} dz] = 0.$$

- Motivated by the explicit form of the bosonic Lagrangian we will study more general setting. Let $\varphi = (\varphi_1, \dots, \varphi_{N-1})$ be the $N-1$ component bosonic field and consider the theory

$$\mathcal{L} = \frac{1}{8\pi} (\partial_\mu \varphi)^2 + \Lambda \sum_{r=1}^N e^{(\alpha_r, \varphi)},$$

where $(\alpha_1, \dots, \alpha_N)$ is a given set of vectors, which is required to have maximal rank.

- We are only interested in the quantum field theories of this form with infinitely many integrals of motion. In the leading order in Λ this constraints the integrals of motion $\mathbf{I}_k^{\text{free}}$ to obey

$$[\mathbf{I}_k^{\text{free}}, \int e^{(\alpha_r, \varphi)} dz] = 0, \quad \text{for all } r = 1, \dots, N.$$

- Practically, we study the equations

$$\oint_{\mathcal{C}_z} e^{(\mathbf{a}_r, \varphi(\xi))} W_{k-1}(z) d\xi = \partial(\dots),$$

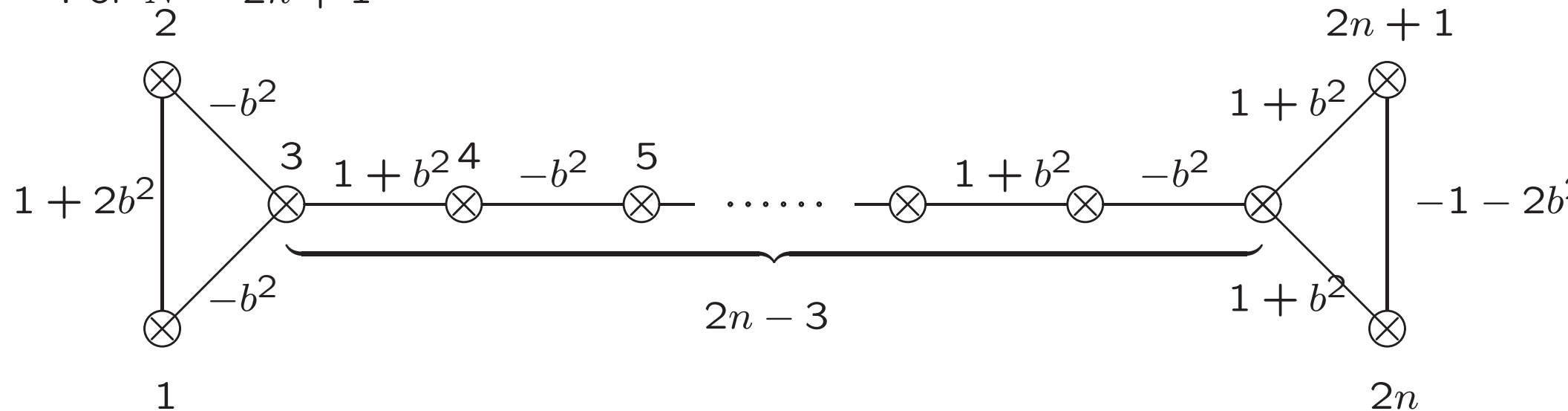
where $W_k(z)$ is some preferred set of currents in W -algebra. This implies that the charges

$$\mathbf{I}_{k-1} = \int W_k(z) dz,$$

commute with all exponentials.

- There many solutions. Important for us, correspond to fermionic screenings: $(\alpha, \alpha) = -1$

For $\frac{N}{2} = 2n + 1$



This picture defines an integrable system $[\mathbf{I}_{2k-1}, \mathbf{I}_{2l-1}] = 0$ with

$$\mathbf{I}_{2k-1} = \int W_{2k}(x) dx.$$

Explicitly one has $\varphi = (\Phi_1, \dots, \Phi_n, \phi_1, \dots, \phi_n)$

$$W_2(x) = (\partial\Phi, \partial\Phi) + (\partial\phi, \partial\phi),$$

and

$$\begin{aligned} W_4(x) &= ((\partial\Phi, \partial\Phi) + (\partial\phi, \partial\phi))^2 + \frac{2n-1}{3} \sum_{k=1}^n \left(\frac{1}{b^2} (\partial\Phi_k)^4 + \frac{1}{a^2} (\partial\phi_k)^4 \right) + \\ &+ 2(2n-1) \sum_{k=1}^n \left((\partial\Phi_k)^2 + (\partial\phi_k)^2 \right) \left(\frac{1}{a} \sum_{j>k} \partial^2\Phi_j + \frac{1}{b} \sum_{j<k} \partial^2\phi_j - \frac{2}{2n-1} \sum_{j=1}^n (j-1) \left(\frac{1}{a} \partial^2\Phi_j + \frac{1}{b} \partial^2\phi_j \right) \right) \\ &+ \left(\frac{4(n+1)}{3} + \frac{2n-1}{3} \left(\frac{1}{b^2} + \frac{2}{a^2} \right) \right) (\partial^2\Phi, \partial^2\Phi) + \left(\frac{4(n+1)}{3} + \frac{2n-1}{3} \left(\frac{2}{b^2} + \frac{1}{a^2} \right) \right) (\partial^2\phi, \partial^2\phi) \\ &+ 2 \sum_{i \leq j} (i-1)(2(j-n)-1)(2-\delta_{ij}) \left(\frac{1}{a^2} \partial^2\Phi_i \partial^2\Phi_j + \frac{1}{b^2} \partial^2\phi_{n-i+1} \partial^2\phi_{n-j+1} \right) + \\ &+ \frac{2}{ab} \left(4 \sum_{i,j} (i-1)(n-j) \partial^2\Phi_i \partial^2\phi_j - (2n-1) \sum_{i>j} (2(i-j)-1) \partial^2\Phi_i \partial^2\phi_j \right). \end{aligned}$$

- Using boson-fermion correspondence we will arrive to the Lagrangian which is supposed to describe the theory

$$\begin{aligned}
\mathcal{L} = & \sum_{k=1}^n \left(\frac{1}{8\pi} (\partial_\mu \Phi_k)^2 + i\bar{\psi}_k \gamma^\mu \partial_\mu \psi_k + \frac{\pi b^2}{2(1+b^2)} (\bar{\psi}_k \gamma^\mu \psi_k)^2 \right) - \\
& - m \left(e^{b\Phi_1} \bar{\psi}_1 \psi_1 + \sum_{k=2}^{n-1} \left(e^{b\Phi_k} \bar{\psi}_k \left(\frac{1+\gamma_5}{2} \right) \psi_k + \right. \right. \\
& + e^{-b\Phi_{k-1}} \bar{\psi}_k \left(\frac{1-\gamma_5}{2} \right) \psi_k \left. \right) + e^{-b\Phi_{n-1}} \bar{\psi}_n \left(\frac{1-\gamma_5}{2} \right) \psi_n + \\
& \left. + \cosh b\Phi_n \bar{\psi}_n \left(\frac{1+\gamma_5}{2} \right) \psi_n \right) - \\
& - \frac{m^2}{8\pi b^2} \left(e^{2b\Phi_1} + 2 \sum_{k=2}^{n-1} e^{b(\Phi_k - \Phi_{k-1})} + e^{b(\Phi_n - \Phi_{n-1})} + e^{-b(\Phi_{n-1} + \Phi_n)} \right).
\end{aligned}$$

- Last term corresponds to the counter term.

One can show that in the theories with fermionic screenings there are also screenings of sigma-model type. It means that for any two fermionic screenings with exponents α_i and α_j such that $(\alpha_i, \alpha_j) \neq 0$ there is also the field

$$\mathcal{V}_{i,j} = (\alpha_i, \partial\varphi) e^{(\beta_{ij}, \varphi)}, \quad \text{where} \quad \beta_{ij} = \frac{2}{(\alpha_i + \alpha_j)^2} (\alpha_i + \alpha_j),$$

such that

$$\oint_{\mathcal{C}_z} W_{2k}(z) \mathcal{V}_{i,j}(w) dw = \partial_z \mathcal{V}_k^{(i,j)}(z).$$

This fact suggests the idea to consider the theory

$$\mathcal{L} = \frac{1}{8\pi} (\partial_\mu \varphi)^2 + \mu \sum_{(i,j) \in \mathbf{I}} (\alpha_i, \partial\varphi) (\alpha_i, \bar{\partial}\varphi) e^{(\beta_{ij}, \varphi)} + \dots,$$

which might be the dual sigma-model description of the original theory.

By ... we denoted possible counterterms. In order to take them into account we will use RG approach.

- Within this approach we have to study the one-loop evolution equation

$$R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Psi = -\dot{G}_{\mu\nu},$$

Namely, we are looking for the solution to this equation with the UV asymptotic prescribed by the bare Lagrangian

$$\mathcal{L} = \frac{1}{8\pi} (\partial_\mu \varphi)^2 + \mu \sum_{(i,j) \in \mathbf{I}} (\alpha_i, \partial\varphi)(\alpha_i, \bar{\partial}\varphi) e^{(\beta_{ij}, \varphi)} + \dots,$$

- Here Ψ describes the effect of t -dependent diffeomorphisms (not all).

Consider the simplest case: $N = 5$. Let, (x_1, x_2, x_3, x_4) be the local coordinates in \mathbb{R}^4 . We expect that the solution should behave in UV $t \rightarrow -\infty$ as

$$G_{\mu\nu} = \delta_{\mu\nu} + e^{\alpha t} \left(A_{\mu\nu} e^{x_1} + B_{\mu\nu} e^{-x_1 - x_2} + C_{\mu\nu} e^{-x_1 + x_2} \right) + \dots,$$

$$\Psi = (\rho, x) + \dots,$$

where

$$A_{\mu\nu} = \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{pmatrix}, \quad C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{pmatrix},$$

and ρ , some unknown constant vector.

At leading order we have

$$\alpha = \frac{3}{4}, \quad \rho = \left(-\frac{1}{4}, 0, -\frac{3i}{4}, \frac{i}{2} \right).$$

Solving the asymptotic problem one can see that

$$G_{\mu\nu} = \begin{pmatrix} F_1 & 0 & iF_5 & 0 \\ 0 & F_2 - \cosh(x_2)F_6 & 0 & -i \sinh(x_2)F_6 \\ iF_5 & 0 & F_3 & 0 \\ 0 & -i \sinh(x_2)F_6 & 0 & F_4 + \cosh(x_2)F_6 \end{pmatrix}, \quad \Psi = (\rho, x) + F_7.$$

These equations are compatible provided that

$$F_2 = F_4, \quad F_2^2 = 1 + F_6^2.$$

The function F_7 is arbitrary. We can choose it such that the following relations holds

$$\det G = F_1 F_3 + F_5^2 = 1.$$

With this choice the problem has the unique solution

$$F_3 = \frac{(1-U)(1-UV)}{1-U^2V}, \quad F_5 = \frac{U(1-V)}{1-U^2V} + \frac{2UV(1-U)}{3(1-U^2V)},$$

$$F_6 = V^{\frac{1}{2}} \left(\frac{(1+V)(1+U^2V)}{(1-V)^2(1-U^2V)} - \left(\frac{1}{2} + \frac{4V}{(1-V)^2} \right) \frac{U}{1-U^2V} \right),$$

$$F_7 = \log \left(\frac{(1-UV)^2}{1-V} \right)$$

where

$$U = e^{\frac{3t}{4}} e^{x_1}, \quad V = \frac{1}{4} e^{\frac{3t}{2}} e^{-2x_1}.$$

It is convenient to introduce new coordinates ζ , θ , ϕ_1 and ϕ_2 by the following equations

$$F_3 = \frac{\kappa(1 - \zeta^2)}{(1 - \kappa^2\zeta^2)}, \quad \tanh\left(\frac{x_2}{2}\right) = \sin\theta,$$

$$\kappa = \frac{2 - e^{\frac{3t}{2}}}{2 + e^{\frac{3t}{2}}}, \quad \phi_1 = \frac{x_3}{2}, \quad \phi_2 = \frac{x_4}{2} - \frac{i}{2} \log \cos\theta.$$

In these coordinates the metric has the form (after rescaling $ds^2 \rightarrow 4\nu ds^2$, $t \rightarrow 4\nu t + \log 2$)

$$ds^2 = \frac{\kappa}{\nu} \left(\frac{d\zeta^2}{(1 - \zeta^2)(1 - \kappa^2\zeta^2)} + \frac{(1 - \zeta^2)d\phi_1^2}{(1 - \kappa^2\zeta^2)} + \zeta^2 d\theta^2 + 2i\zeta^2 \tan\theta d\theta d\phi_2 + \frac{(1 - \kappa^2\zeta^4 \sin^2\theta)d\phi_2^2}{\kappa^2\zeta^2 \cos^2\theta} \right).$$

We can also perform the T-duality in the ϕ_2 isometry making the metric diagonal

$$d\tilde{s}^2 = \frac{\kappa}{\nu} \left[\frac{d\zeta^2}{(1 - \zeta^2)(1 - \kappa^2\zeta^2)} + \frac{(1 - \zeta^2)d\phi_1^2}{(1 - \kappa^2\zeta^2)} + \frac{\zeta^2}{1 - \kappa^2\zeta^4 \sin^2 \theta} (d\theta^2 + \cos^2 \theta d\phi_2^2) \right]$$

and generating the non-zero pure imaginary B -field

$$B = \frac{i\kappa^2 \sin \theta \cos \theta \zeta^4}{\nu(1 - \kappa^2\zeta^4 \sin^2 \theta)} d\theta \wedge d\phi_2.$$

It can be checked that this metric and B -field coincide with the ones obtained from the $\eta = i\kappa$ -deformed $O(5)$ sigma-model action mentioned in the introduction.

Conclusions

- Presumably, the duality should happen for all η -deformed SSSM. It is not known how to arrive to the system of screening fields starting from the deformed sigma-model action.
- At the moment it is unclear how to consider supergroup valued sigma-models in this context (important for AdS/CFT sigma-models).
- Once the dual theory is constructed. One has a system of local IM's. It is interesting to consider spectral problem for this system.
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