

Homotopy operators. Properties and application in HS equations

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4 June 2018

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Introduction

Dynamic HS equations schematically

$$\begin{aligned}d_x \omega + \omega * \omega &= \Upsilon(\omega, \omega, C) + \Upsilon(\omega, \omega, C, C) + \dots, \\d_x C + [\omega, C]_* &= \Upsilon(\omega, C, C) + \dots\end{aligned}$$

$\omega = dx^\mu \omega_\mu(Y, K|x)$ – generating function for HS gauge fields.

$C = C(Y, K|x)$ – generating function for HS stress-tensors (Maxwell, Weyl, etc) + scalar $C(0, K|x)$.

$\Upsilon(\omega, \omega, C, C), \Upsilon(\omega, C, C), \dots$ – might contain infinite number of derivatives between C -fields.

Υ -vertices can be obtained from generating system with doubled amount of spinorial variables

Introduction

Vasiliev equations in $d = 4$

$$\begin{aligned}d_x W + W * W &= 0, \\d_x S + [W, S]_* &= 0, \\d_x B + [W, B]_* &= 0, \\S * S &= i(\theta^A \theta_A + \eta B * \gamma + \bar{\eta} B * \bar{\gamma}), \\[S, B]_* &= 0.\end{aligned}$$

Here $W = dx^\mu W_\mu(Z, Y, K)$, $S = \theta^A S_A(Z, Y, K)$, $B = B(Z, Y, K)$.

$Z_A = (z_\alpha, \bar{z}_{\dot{\alpha}})$, $Y_A = (y_\alpha, \bar{y}_{\dot{\alpha}})$, $\theta^A = (\theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, $K = (k, \bar{k})$

$$\{y_\alpha, k\} = \{z_\alpha, k\} = \{\theta^\alpha, k\} = 0, \quad [\bar{y}_{\dot{\alpha}}, k] = [\bar{z}_{\dot{\alpha}}, k] = [\bar{\theta}^{\dot{\alpha}}, k] = 0, \quad k^2 = 1.$$

$$d_x = dx^\mu \frac{\partial}{\partial x^\mu}, \quad \{\theta^A, \theta^B\} = \{dx^\mu, \theta^A\} = 0 \implies \left\{ \frac{\partial}{\partial \theta^A}, \theta^B \right\} = \left\{ \frac{\partial}{\partial \theta^A}, dx^\mu \right\} = 0.$$

Introduction

Star-product

$$(f * g)(Z, Y) = \frac{1}{(2\pi)^4} \int dU dV f(Z + U, Y + U) g(Z - V, Y + V) e^{iU_A V^A}.$$

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2i\epsilon_{AB}, \quad [Y_A, Z_B]_* = 0.$$

Central elements $\gamma, \bar{\gamma}$

$$\gamma = e^{iz_\alpha y^\alpha} k \theta^\alpha \theta_\alpha, \quad \bar{\gamma} = e^{i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}} \bar{k} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}}$$

Inner Klein operators

$$f(z, \bar{z}, y, \bar{y}) * e^{iz_\alpha y^\alpha} = e^{iz_\alpha y^\alpha} * f(-z, \bar{z}, -y, \bar{y}).$$

Introduction

Perturbative expansion

$$B_0 = 0, \quad S_0 = \theta^A Z_A, \quad W_0 = \omega(y, \bar{y}).$$

Here $\omega(y, \bar{y})$ satisfies $d_x \omega + \omega * \omega = 0$.

$$[S, B]_* = 0 \implies [S_0, B_1]_* + [S_1, B_0]_* = 0,$$

Using star-product $[Z_A, f(Z, Y)]_* = -2i \frac{\partial}{\partial Z^A} f(Z, Y)$

$$[S_0, B_1] = -2i \theta^A \frac{\partial}{\partial Z^A} B_1(Z, Y, K) = -2i d_Z B_1(Z, Y, K) = 0.$$

$$B_1(Z, Y, K) = C(Y, K).$$

Introduction

$$-2id_Z S_1 = i\eta C * \gamma + i\bar{\eta} C * \bar{\gamma}$$

Typical equation

$$d_Z f(Z, Y|\theta) = J(Z, Y|\theta)$$

Solution up to d_Z -exact terms is given by homotopy operator Δ_Q

$$f(Z, Y|\theta) = \Delta_Q J(Z, Y|\theta),$$

where

$$\Delta_Q J(Z, Y|\theta) \equiv (Z + Q)^A \frac{\partial}{\partial \theta^A} \int_0^1 \frac{dt}{t} J(tZ - (1-t)Q, Y|t\theta).$$

Where Q should be Z -independent. When $Q = 0$, Δ_0 – conventional homotopy.

Homotopy operator Δ_Q

$$\Delta_Q J(Z, Y|\theta) \equiv (Z + Q)^A \frac{\partial}{\partial \theta^A} \int_0^1 \frac{dt}{t} J(tZ - (1-t)Q, Y|t\theta).$$

How to obtain this formula?

From Z to $Z'_A = Z_A + Q_A$

$$J(Z, Y|\theta) \implies J(Z' - Q, Y|\theta)$$

Conventional homotopy over Z'

$$Z'^A \frac{\partial}{\partial \theta^A} \int_0^1 \frac{dt}{t} J(tZ' - Q, Y|t\theta).$$

Homotopy operator properties

Index A in (Z_A, Y_A, θ^A) takes arbitrary values

Resolution of identity

$$d_Z \Delta_Q + \Delta_Q d_Z + h_Q = 1, \quad \text{where} \quad h_Q f(Z, Y|\theta) \equiv f(-Q, Y|0)$$

Using definition and resolution of identity one can show

$$\Delta_Q \Delta_P + \Delta_P \Delta_Q = 0 \implies \Delta_P \Delta_P = 0,$$

$$h_P \Delta_Q = -h_Q \Delta_P \implies h_P \Delta_P = 0,$$

$$\Delta_B - \Delta_A = \{d_Z, \Delta_A \Delta_B\} + h_A \Delta_B$$

Moreover

$$\Delta(\rho) := \int dQ \rho(Q) \Delta_Q, \quad \text{with} \quad \int dQ \rho(Q) = 1.$$

Homotopy operator properties

Two component relations $(Z_A, Y_A, \theta^A) \implies (z_\alpha, y_\alpha, \theta^\alpha)$

$$\Delta_b \Delta_a f(z, y) \theta^\beta \theta_\beta = 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (z+b)_\alpha (z+a)^\alpha f(\tau_1 z - \tau_3 b - \tau_2 a, y)$$

$$h_c \Delta_b \Delta_a f(z, y) \theta^\beta \theta_\beta = 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (b-c)_\gamma (a-c)^\gamma f(-\tau_1 c - \tau_3 b - \tau_2 a, y).$$

As consequence

$$h_{(\mu+1)q_2 - \mu q_1} \Delta_{q_2} \Delta_{q_1} = 0 \quad \forall \mu \in \mathbb{C}$$

$$\Delta_b \Delta_a \gamma = 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (z+b)_\alpha (z+a)^\alpha e^{i(\tau_1 z - \tau_2 a - \tau_3 b)_\beta y^\beta} k,$$

$$h_c \Delta_b \Delta_a \gamma = 2 \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) (b-c)_\beta (a-c)^\beta e^{-i(\tau_1 c + \tau_2 a + \tau_3 b)_\alpha y^\alpha} k.$$

As consequence

$$h_{a+\alpha y} \Delta_{b+\alpha y} \Delta_{c+\alpha y} \gamma = h_a \Delta_b \Delta_c \gamma \quad \forall \alpha \in \mathbb{C}.$$

Homotopy operator properties

$$(\Delta_d - \Delta_c)(\Delta_a - \Delta_b)\gamma = (h_d - h_c)\Delta_a\Delta_b\gamma,$$

Star-exchange properties

$$\Delta_{q+\alpha y}(C(y, k) * \Gamma(z, y|\theta)) = C(y, k) * \Delta_{q+(1-\alpha)p+\alpha y}\Gamma(z, y|\theta),$$

where q is z and y independent spinor.

$$p_\alpha = -i\partial_\alpha^C, \quad p_\alpha C(y, k) \equiv C(y, k)p_\alpha = -i\frac{\partial}{\partial y^\alpha}C(y, k)$$

$$\Delta_{q+\alpha y}(\Gamma(z, y; \theta) * C(y, k)) = \Delta_{q+(1+\alpha)p+\alpha y}\Gamma(z, y; \theta) * C(y, k)$$

$$\Delta_{q+\alpha y}\gamma * C(y, k) = C(y, k) * \Delta_{q+2p+\alpha y}\gamma.$$

Central on-mass-shell theorem. Preliminaries

Only holomorphic sector

$$-2id_z S_1 = i\eta C * \gamma \implies S_1 = -\frac{\eta}{2} \Delta_0 (C * \gamma) = -\frac{\eta}{2} C * \Delta_p (\gamma),$$

where $d_z = \theta^\alpha \frac{\partial}{\partial z^\alpha}$.

$$2id_z W_1 = d_x S_1 + [\omega, S_1]_* \implies W_1 = \frac{1}{2i} \Delta_0 (d_x S_1 + [\omega, S_1]_*)$$

$$W_1 = -\frac{\eta}{4i} (C * \omega * \Delta_{p+t} \Delta_{p+2t} \gamma - \omega * C * \Delta_{p+t} \Delta_p \gamma),$$

where

$$t_\alpha = -i\partial_\alpha^\omega, \quad t_\alpha \omega(y) \equiv \omega(y) t_\alpha = -i \frac{\partial}{\partial y^\alpha} \omega(y).$$

Thanks to star-exchange formulas!

Central on-mass-shell theorem. Dynamic equations

$$d_x \omega + \omega * \omega = - \underbrace{d_x W_1}_{\text{proportional to } z_\alpha} - \omega * W_1 - W_1 * \omega$$

$d_x W_1$ is proportional to z_α .

$$d_x \omega + \omega * \omega = 0, \quad d_x C + \omega * C - C * \omega = 0.$$

$$d_x (\omega * C * \Delta_{p+t} \Delta_p \gamma) = -\omega * \omega * C * \Delta_{p+t_1+t_2} \Delta_p \gamma + \dots$$

$$\begin{aligned} d_x \omega + \omega * \omega &= \frac{\eta}{4i} \omega * \omega * C * (\Delta_{p+t_1+t_2} - \Delta_{p+t_2}) (\Delta_p - \Delta_{p+t_2}) \gamma + \\ &+ \frac{\eta}{4i} \omega * C * \omega * (\Delta_{p+t_1+t_2} - \Delta_{p+2t_2}) (\Delta_{p+t_2} - \Delta_{p+t_1+2t_2}) \gamma + \\ &+ \frac{\eta}{4i} C * \omega * \omega * (\Delta_{p+t_1+t_2} - \Delta_{p+t_1+2t_2}) (\Delta_{p+t_1+2t_2} - \Delta_{p+2t_1+2t_2}) \gamma. \end{aligned}$$

Recall the following property

$$(\Delta_d - \Delta_c)(\Delta_a - \Delta_b)\gamma = (h_d - h_c)\Delta_a\Delta_b\gamma.$$

Central on-mass-shell theorem. Dynamic equations

$$d_x \omega + \omega * \omega = \Upsilon_{\omega\omega C} + \Upsilon_{\omega C\omega} + \Upsilon_{C\omega\omega}$$

$$\begin{aligned} \Upsilon_{\omega\omega C} &= \frac{\eta}{4i} \omega * \omega * C * h_{p+t_1+t_2} \Delta_p \Delta_{p+t_2} \gamma = \\ &= \frac{\eta}{2i} \int_{[0,1]^3} d^3 \tau \delta(1 - \tau_1 - \tau_2 - \tau_3) e^{i(1-\tau_3)\partial_1^\alpha \partial_{2\alpha}} \times \\ &\quad \times \partial^\alpha \omega((1 - \tau_1)y, \bar{y}) * \partial_\alpha \omega(\tau_2 y, \bar{y}) * C(\underbrace{-\tau_1 \partial_1 - (1 - \tau_2) \partial_2}_{}, \bar{y}; K) k \end{aligned}$$

No y -dependence in C ! Ultra-local

Conclusion

- Background independent central on-mass-shell theorem with ultra-local structure of vertices
- Thanks to star-exchange formulas structures $\Delta\gamma$, $\Delta\Delta\gamma$, $\Delta(\Delta\gamma * \Delta\gamma)$, etc. appeared to be the only thing to be analyzed
- Generalized homotopy operators may be also applied in $3d$ Prokushkin-Vasiliev theory (work in progress)