# On the structure of conformal higher spin equations


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**based on:**
**M. Grigoriev, A. H. - work in progress**

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# Motivation

- CHS equations (Fradkin-Tseytlin equations)
  - Have a form of $(\Box^{\frac{n-4}{2}+s-t+1} + \dots)\Phi(x,p) = 0$
  - Admit gauge invariance $(\Box^{\frac{n-4}{2}+s} + \dots)\frac{\partial}{\partial p}\nabla\Phi(x,p) = 0$ (shown for depth 1 CHS fields)
  - CHS operators factorise in a product of second order operators (gauge fixed version Tseytlin 2013], earlier related works: [Metsaev 07], [Joung Mkrtchyan 12], [Deser, Joung, Waldron 12], [Gover 06]
  - Factorisation on the lagrangian level [Metsaev 2014]
  - Explicit formulas for factorised operator were proposed by [Nutma, Taronna 2015]
  - More general fields[Vasiliev 09]

# Motivation

- GJMS operators
  - Defined on densities or tractor tensor fields
  - Have a form of $\Box^k + \ldots$
  - Factorise on the Einstein background[Gover, 06] in a product of second-order operators.

The goal is to study the structure of CHS operators; to find a similar to the GJMS case generating procedure for the CHS operators and (generalised) higher-depth CHS operators and investigate compatibility of the gauge transformation with the factorisation

# Flat model for conformal geometry

An ambient space is $\mathbb{R}^{n,2}$.
$X^A (A = +, -, 0, 1, ..., n-1)$
Equipped with metric
$\eta_{AB} : \eta_{+-} = \eta_{-+} = 1, \;\; \eta_{ab} = diag(-1, 1..., 1), \;\; a, b = 0...n-1.$ Denote $B \cdot C = B_A C^A$

$n$-dimensional conformal space $M$:

- Quotient space of a cone $X^2 = 0$ modulo the equivalence relation $X^A \sim \lambda X^A$
- Equipped with the conformal structure inherited from the ambient metric $\eta_{AB}$
- $O(n, 2)$ acts by conformal isometries

# Tractors in ambient picture

Totally symmetric fields on the ambient space can be written in terms of generating functions

$$\Phi(X,P) = \sum_{i=0} \Phi^{A_1 \dots A_s} P_{A_1} \dots P_{A_s} \, .$$

One can define $\Phi$ on the conformal space $M$ and extend it to the ambient space:

$$(X \cdot \frac{\partial}{\partial X} - w)\Phi(X,P) = 0$$

$$\Phi(X) \sim \Phi(X) + X^2 \chi(X,P)$$

This defines $\mathcal{E}^\bullet[w]$.
$\mathcal{E}^\bullet[w]$ is a space of symmetric tractor tensors of weight $w$.
(curved case -[Cap, Gover 02])

# Tractors in the parent approach

[M. Grigoriev, A. Waldron 2011]:

- Any conformally-flat manifold M
- $\Phi$ is manifestly defined on $M$, not the ambient space

$$\nabla_\mu \Phi = 0, \qquad (\nabla^2 = 0)$$
$$((Y+V)\frac{\partial}{\partial Y} - w)\Phi = 0$$
$$\Phi \sim \Phi + (Y+V)^2 \chi$$

- Manifestly covariant - one can use general local coordinates on M

Also denote the space of all solutions by $\mathcal{E}^\bullet[w]$. After the elimination of auxiliary variables $Y$ we end up with tractors.

The covariant derivative and the compensator are:

$$\nabla_\mu = \partial_\mu - \omega_\mu{}^A{}_B((Y^B + V^B)\frac{\partial}{\partial Y^A} + P^B \frac{\partial}{\partial P^A}), \quad V^A = \begin{pmatrix} V^+ \\ V^a \\ V^- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

# GJMS operators in the parent approach

- Equivariant with respect to o(n, 2) (i.e. commutes with o(n,2))
- Well-defined on tractors:

$$\nabla_\mu \Phi = 0$$
$$((Y+V)\frac{\partial}{\partial Y} - w)\Phi = 0$$
$$\Phi \sim \Phi + (Y+V)^2 \chi$$

Consider operator $\Delta^k = (\frac{\partial}{\partial Y}\frac{\partial}{\partial Y})^k$ acting on $\mathcal{E}^\bullet[k - \frac{n}{2}]$. It is well-defined on the equivalence classes [Graham,.. 1992]

$$\Delta^k((Y+V)^2\chi) = (Y+V)^2\Delta^k\chi + 4k\Delta^{k-1}(w_\chi + \frac{n}{2} - k + 2)\chi = (Y+V)^2\Delta^k\chi$$

And thus descends to conformally invariant operator on $M$ denoted by $P^{2k}$[Paneitz 1983, Fradkin, Tseytlin 1982]

# Thomas-D operator

$$\mathcal{D}_A := 2((Y+V)\cdot\frac{\partial}{\partial Y}+\frac{n}{2})\frac{\partial}{\partial Y^A}-(Y+V)_A\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial Y}$$

Well-defined on $\mathcal{E}^{\bullet}[w]$:

- Preserves the equivalence relation $\Phi \sim \Phi + (Y+V)^2\chi$
- Lowers conformal weight by 1: $P^A\mathcal{D}_A : \mathcal{E}^{\bullet}[w] \mapsto \mathcal{E}^{\bullet}[w-1]$

Relation between $\mathcal{D}$ and $\Delta^k$:

- While acting on $\Phi \in \mathcal{E}^{\bullet}[k-\frac{n}{2}]$:
  $\mathcal{D}_{A_1}...\mathcal{D}_{A_s}\Phi = (-1)^k(Y+V)_{A_1}...(Y+V)_{A_s}\Delta^k\Phi$ [Gover, Peterson 03]

# Factorisation of GJMS operators

Scale tractor $I \in \mathcal{E}^1[0]$:

- $\nabla_\mu I^A = 0$
- on (A)dS $I^A = \begin{pmatrix} 1 \\ 0 \\ -\frac{J}{n} \end{pmatrix}$
- $I \cdot \mathcal{D} : \mathcal{E}^\bullet[w] \mapsto \mathcal{E}^\bullet[w-1]$
- On (A)dS $I \cdot V = 1$
- $I^A$ breaks O(n, 2) symmetry down to (A)dS symmetry

The relation between $\mathcal{D}$ and $\Delta^k$ imply [Gover, 06]:

$$I^{A_1}...I^{A_k}\mathcal{D}_{A_1}...\mathcal{D}_{A_k} = (-1)^k I^{A_1}...I^{A_k}(Y+V)_{A_1}...(Y+V)_{A_k}\Delta^k = \\ = (-1)^k\Delta^k = (I \cdot \mathcal{D})^k$$

Note that $\Delta^k$ change the weight by $2k$, but $(I \cdot \mathcal{D})^k$ only by $k$.

Who is factorised?

$I \cdot \mathcal{D}$ respects the equivalence relation, but $\Delta$ does not. $\Rightarrow$ $(I \cdot \mathcal{D})^k$ factorise.

# CHS fields, manifestly O(n,2) description

[Bekaert, Grigoriev 2013]

$$\nabla\Phi = 0,\ \ ((Y+V)\frac{\partial}{\partial Y} - w)\Phi = 0,\ \ (Y+V)\frac{\partial}{\partial P}\Phi = 0$$
$$(\frac{\partial}{\partial Y}\frac{\partial}{\partial Y})^k\Phi = 0,\ \ \frac{\partial}{\partial Y}\frac{\partial}{\partial P}\Phi = 0$$
$$\frac{\partial}{\partial P}\frac{\partial}{\partial P}\Phi = 0,\ \ P\frac{\partial}{\partial P}\Phi = s\Phi$$

Gauge transformations: $\Phi \sim \Phi + (P\frac{\partial}{\partial Y})^t\chi$

Weight $w = s - t - 1$, $t = 1, 2, ... \frac{n-4}{2} + s$ (the lowest is $1 - \frac{n}{2}$, the highest $s - 2$)

- originates from AdS system for PM fields [Alkalaev, Grigoriev 2011]
- CHS fields arise as boundary values of corresponding PM fields
- The system encodes several equations, one of them is a CHS equation, but it does not have a manifestly factorised form.

# Manifestly O(n, 2) formulation of off-shell CHS fields

For our purpose consider the following off-shell system:

- For every off-shell CHS field there is a unique equivalence class in this system and v. v.

$$\nabla\Phi = 0, \ \ ((Y+V)\frac{\partial}{\partial Y} - w)\Phi = 0, \ \ (Y+V)\frac{\partial}{\partial P}\Phi = 0$$

$$\Phi \sim \Phi + (Y+V)^2\chi, \ \ \mathcal{D}\frac{\partial}{\partial P}\Phi = 0$$

$$\frac{\partial}{\partial P}\frac{\partial}{\partial P}\Phi = 0, \ \ P\frac{\partial}{\partial P}\Phi = s\Phi$$

With gauge transformations: $\Phi \sim \Phi + (P\mathcal{D})^t\chi$

We'll denote this system $S[s, w]$

# On-shell CHS system

Introduce an analogue of $I \cdot \mathcal{D}$:
A well-defined on $S[s,w]$ operator $B : S[s,w] \mapsto S[s,w-1]$.

$$B := I \cdot \mathcal{D} - \frac{1}{s-1-w} P \cdot \mathcal{D} \;\; I \cdot \frac{\partial}{\partial P}$$

May we expect that powers of $B$ coincide with CHS operator as well as powers of $I\mathcal{D}$ coincide with $\Delta^k$?
**Yes**

$$B^{\frac{n-4}{2}+s-t+1}\Phi = 0$$

is a depth-t CHS equation, $\Phi \in S[s, s-t-1]$
Gauge invariance: $B^{\frac{n-4}{2}+s-t+1}(P\mathcal{D})^t \chi = 0$

# Factorisation of CHS operators in the ordinary notation

- Use a lift $T_w$ from CHS field $\Phi(x,p)$ of weight $w$ to $\Phi(x,P,Y) \in S[s,w]$ and its inverse.
- Denote $A_w := T^{-1}_{w-1} B T_w$. $A_w$ maps CHS fields of weight $w$ to CHS fields of weight $w-1$
- The formula for CHS operator takes the following form:

$$T^{-1}_{w-k} B^k T_w \phi(x,p) = A_{w-k+1} A_{w-k+2} ... A_w \phi(x,p), \quad w = k - \frac{n}{2}$$

- The explicit formula for $T^{-1}_{w-1} B T_w$:

$$T^{-1}_{w-1} B T_w \Phi(x,p) = \{\Box^0 + \frac{2J}{n}(-s + (n+w-1)w) - \\ - \frac{n+2s-4}{(s-1-w)(n+s+w-2)}(p\nabla^0)(\frac{\partial}{\partial p}\nabla^0) + \\ + \frac{1}{(n+s+w-2)(s-1-w)} p^2 (\frac{\partial}{\partial p}\nabla^0)^2\} \Phi(x,p)$$

Once we found $T_{w-1}^{-1} B T_w$ it is easy to write down a factorised form of CHS operator:

$$FT_{s,t}\Phi(x,p) = \prod_{i=0}^{\frac{n-4}{2}+s-t} \{\Box^0 + \frac{2J}{n}(-s+(n+w-i-1)(w-i))-$$
$$-\frac{n+2s-4}{(s-1-w+i)(n+s+w-i-2)}(p\nabla^0)(\frac{\partial}{\partial p}\nabla^0)+$$
$$+\frac{1}{(n+s+w-i-2)(s-1-w+i)}p^2(\frac{\partial}{\partial p}\nabla^0)^2\}\Phi(x,p)$$

At $t=1$ this reproduces the formula obtained by Nutma and Taronna 2015

# Example: conformal graviton

For $s = 2, t = 1, n = 4$. We have an equation for conformal graviton

$$[\Box^0 - 2J - \frac{2}{3}(p\nabla^0)(\frac{\partial}{\partial p}\nabla^0) + \frac{1}{6}p^2(\frac{\partial}{\partial p}\nabla^0)^2][\Box^0 - J -$$
$$-(p\nabla^0)(\frac{\partial}{\partial p}\nabla^0) + \frac{1}{4}p^2(\frac{\partial}{\partial p}\nabla^0)^2]\phi(x,p) = 0$$

The second operator is a CHS operator of depth 2, while the first one is a gauge fixed version of Pauli-Fierz operator [Deser, Nepomechie 1984]

# Gauge transformations of CHS operators

- depth-t CHS operator is $B^{\frac{n-4}{2}+s-t+1}$
- Recall the gauge transformations $\delta\Phi \sim (P\cdot\mathcal{D})^t\chi$, $\chi$ has weight $w_\chi = s-1$, $s_\chi = s-t$.

•Observe that $B(P\cdot\mathcal{D})^t\chi = (-1)(P\cdot\mathcal{D})^{t+1} I\cdot\frac{\partial}{\partial P}\chi$
•Apply again

$$(P\cdot\mathcal{D})^{t+2}(I\cdot\frac{\partial}{\partial P})^2\chi$$

The procedure will give zero after $s-t+1$ iterations. $\Rightarrow (P\cdot\mathcal{D})^t\chi$ is indeed in the kernel of a depth-t CHS operator.

$$B(P\cdot\mathcal{D})^t\chi = 0$$

# Who is $B$?

- The mass term in $T_{w-1}^{-1} B T_w, w = s - t - 1$ coincides with that of PM field spin-s, depth-t
- In general B does not coinside with the kinetic operator of the respective PM field. in case t=1 it is a fronsdal operator in trace-free gauge
- (Some of the) gauge transformations can be easily seen from the properties of $B$:

$$B(P \cdot \mathcal{D})^t \chi = (-1)(P \cdot \mathcal{D})^{t+1} I \cdot \frac{\partial}{\partial P} \chi$$

If $I \cdot \frac{\partial}{\partial P} \chi = 0$(divergence-free parameters in metric notation), then $B(P \cdot \mathcal{D})^t \chi = 0$

# Summary and perspectives

- A generating procedure for factorisation of (higher depth) CHS operators is proposed. The technique is somewhat similar to the Gover's approach to the factorisation of GJMS operators
- A gauge invariance of the $B$ operators entering the factorised form is analysed.
- Application to CHS fields on a curved background? E.g. along the lines of [Nutma, Taronna 15], [Grigoriev, Tseytlin 16]
- Manifestly $o(n, 2)$ description of interacting CHS fields [Segal, 2002]? Relation to HS extended Fefferman-Graham constructions?