Towards Scattering Amplitudes from the Multiparticle Higher-Spin Charges

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Introduction

0. Why amplitudes and HS multiparticle charges may be related?
1. MHV amplitudes
2. Unfolded higher-spin equations
   - rank-1 fields $\mathcal{C}$
   - rank-**r** (multiparticle) fields $\mathcal{C}^{\otimes \mathbf{r}}$

3 On-shell-closed differential forms $\Omega \, : \, \mathrm{d}\Omega|_{\text{unfold. eq-ns}} = 0$.
Conserved charges $\mathcal{Q} = \int_{\Sigma} \Omega$

4 Example of reproducing MHV amplitudes

Why $\mathcal{A}$'s and $\mathcal{Q}$'s may be related?

Scattering amplitudes	**Multiparticle HS charges**
Elements of the gauge-invariant S-matrix parametrized by multi-particle asymptotic (free) states	Gauge-invariant conserved quantities built of tensor products of free higher-spin fields
Constructed from classical on-shell data of free fields: light-like momenta (for external legs) and propagators	Constructed with the aid of classical dynamical equations for free higher-spin fields
Matrix elements of the operator $\exp\left(i\int \mathrm{d}^4x\, L\right)$, rational functions of p	Objects $\int\Omega$ constructed from fields C encoding infinitely many derivatives

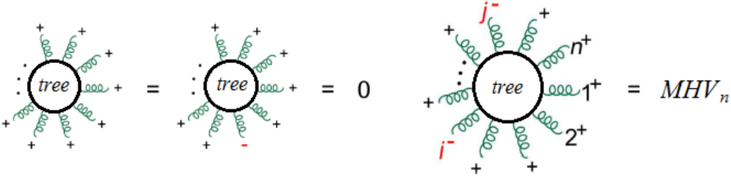
Paradox?

Does a single free classical HS theory enables one to calculate scattering amplitudes of various QFT's?

NO: Free multiparticle HS theory with infinitely many conserved charges is expected to give large enough "kinematical" space, able to contain (reproduce) any scattering amplitude of a given QFT.

MHV amplitudes

- Tree-level color-ordered n-particle amplitudes at the leading order in N in the $SU(N)$ Yang-Mills theory
- One-particle on-shell data is given by light-like momentum $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$, polarization $\epsilon^+_{a\dot{a}} \sim \varkappa_a \widetilde{\lambda}_{\dot{a}}$ or $\epsilon^-_{a\dot{a}} \sim \lambda_a \widetilde{\varkappa}_{\dot{a}}$ (in terms of $SL(2,\mathbb{C})$ spinors) and color.
- For the specific helicity configurations (outgoing particles)



MHV amplitudes

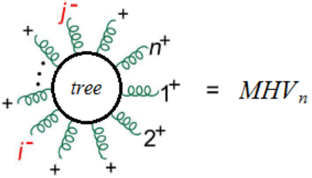
- Expression for MHV_n is given by Parke-Taylor formula (Parke and Taylor '86, proven by Berends, Giele '88)

$$\mathcal{A}\left(1^+_{p_1,c_1}\dots c^-_{p_i,c_i}\dots c^-_{p_j,c_j}\dots c^+_{p_n,c_n}\right) \propto \mathrm{Tr}\left(T^{c_1}\dots T^{c_n}\right)\mathcal{A}^{(ij)}_{\lambda,\widetilde{\lambda}},$$

where

$$\mathcal{A}^{(ij)}_{\lambda,\widetilde{\lambda}} = \frac{\langle\lambda_i\lambda_j\rangle^4}{\langle\lambda_1\lambda_2\rangle\langle\lambda_2\lambda_3\rangle\dots\langle\lambda_n\lambda_1\rangle}\,\delta^{(4)}\Big(\sum_{l=1}^{n}\lambda_l\widetilde{\lambda}_l\Big),$$

with $\langle\lambda_r\lambda_s\rangle = \varepsilon^{ab}\lambda_{ra}\lambda_{sb}$ and $p_{l\,a\dot{a}} = \lambda_{la}\widetilde{\lambda}_{l\dot{a}}$



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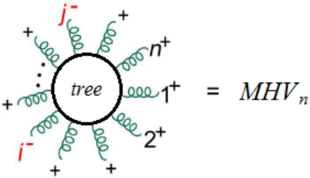

Unfolded HS equations (rank 1)

- rank-1 fields: gauge invariant 0-forms obeying

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} + i\,\frac{\partial^2}{\partial y^a \partial \widetilde{y}^{\dot{a}}}\right) C^1\left(y, \widetilde{y}|x\right) = 0,$$

where $a, \dot{a} = 1, 2$, $x^{a\dot{a}}$ are space-time coordinates, $y^a, \widetilde{y}^{\dot{a}}$ are auxiliary variables.

N.B. Usually formal power series in $y, \widetilde{y}$

$$C^1\left(y, \widetilde{y}|x\right) = \sum_{m,n} \frac{c_{a(m),\dot{a}(n)}(x)}{m!n!}\, y^a \dots y^a\, \widetilde{y}^{\dot{a}} \dots \widetilde{y}^{\dot{a}},$$

unfolded equations encode the set of primary fields $c_{a(n)}(x)$ and $c_{\dot{a}(n)}(x)$ (anti-self-dual and self-dual HS curvatures) and dynamical equations on primaries

$$i\,\frac{\partial}{\partial x^{a\dot{a}}} \sim \mu_a \widetilde{\mu}_{\dot{a}}$$

Unfolded HS equations (rank 1)

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where $a, \dot{a} = 1, 2$, $x^{a\dot{a}}$ are space-time coordinates, $y^a, \widetilde{y}^{\dot{a}}$ are auxiliary variables.

N.B. In what follows $C^1\left(y, \widetilde{y}|x\right)$ is a complex-valued distribution, variables $y, \widetilde{y}$ are involved into play. Plane-wave solutions

$$\exp\left[i\left(\mu_a \widetilde{\mu}_{\dot{a}}\, x^{a\dot{a}} + \mu_a\, y^a + \widetilde{\mu}_{\dot{a}}\, \widetilde{y}^{\dot{a}}\right)\right]$$

obey light-like-momentum constraint

$$i\frac{\partial}{\partial x^{a\dot{a}}} \sim \mu_a \widetilde{\mu}_{\dot{a}}$$

Unfolded HS equations (rank $\mathbf{r}$)

- rank-$\mathbf{r}$ fields: tensor products of $\mathbf{r}$ rank-1 fields obeying

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} + i\,\delta_{IJ}\frac{\partial^2}{\partial y_I^a \partial \widetilde{y}_J^{\dot{a}}}\right) C^{\mathbf{r}}\left(y, \widetilde{y}|x\right) = 0,$$

where $I, J = 1 \ldots \mathbf{r}$. Particular solutions are products

$$C^{\mathbf{r}}\left(y, \widetilde{y}|x\right) = C^1\left(y_1, \widetilde{y}_1|x\right) \ldots C^1\left(y_{\mathbf{r}}, \widetilde{y}_{\mathbf{r}}|x\right),$$

for example plane-wave solutions

$$\chi_{\mu,\widetilde{\mu}} = \exp\left[i\sum_{I=1}^{\mathbf{r}}\left(\mu_{Ia}\,\widetilde{\mu}_{I\dot{a}}\,x^{a\dot{a}} + \mu_{Ia}\,y^{Ia} + \widetilde{\mu}_{I\dot{a}}\,\widetilde{y}^{I\dot{a}}\right)\right]$$

Unfolded HS equations (rank $\mathbf{r}$)

- For a partition $\mathbf{r} = m + \overline{m}$ consider half-Fourier transform

$$y_{1\ldots m}, y_{\underbrace{m+1\ldots\mathbf{r}}_{\overline{m}}} \quad \text{and} \quad \widetilde{y}_{1\ldots\overline{m}}, \widetilde{y}_{\underbrace{\overline{m}+1\ldots\mathbf{r}}_{m}}$$

to

$$\lambda_{1\ldots m}, y_{1\ldots\overline{m}} \quad \text{and} \quad \widetilde{\lambda}_{1\ldots\overline{m}}, \widetilde{y}_{\overline{m}+1\ldots\mathbf{r}}$$

- Rank-$\mathbf{r}$ unfolded equation becomes a first-order PDE

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} - \lambda_{\mathrm{i}\,a}\frac{\partial}{\partial \widetilde{y}^{\dot{a}}_{\mathrm{i}}} - \widetilde{\lambda}_{\bar{\mathrm{i}}\,\dot{a}}\frac{\partial}{\partial y^{a}_{\bar{\mathrm{i}}}}\right) g^{(m,\overline{m})}\left(\lambda_{\mathrm{i}}, y_{\bar{\mathrm{i}}}; \widetilde{y}_{\mathrm{j}}, \widetilde{\lambda}_{\bar{\mathrm{j}}}\middle| x\right) = 0$$

with $\mathrm{i},\mathrm{j} = 1\ldots m$ and $\bar{\mathrm{i}},\bar{\mathrm{j}} = 1\ldots\overline{m}$, with characteristics

$$\lambda_{\mathrm{i}a}, \quad \widetilde{\lambda}_{\bar{\mathrm{i}}\dot{a}}, \quad y^{a}_{\bar{\mathrm{i}}} + x^{a\dot{a}}\,\widetilde{\lambda}_{\bar{\mathrm{i}}\dot{a}}, \quad \widetilde{y}^{\dot{a}}_{\mathrm{i}} + x^{a\dot{a}}\,\lambda_{\mathrm{i}a}$$

Unfolded HS equations (rank r)

- Rank-**r** unfolded equation as a first-order PDE

$$\left(\frac{\partial}{\partial x^{a\dot{a}}} - \lambda_{\mathrm{i}\,a}\frac{\partial}{\partial \widetilde{y}^{\dot{a}}_{\mathrm{i}}} - \widetilde{\lambda}_{\bar{\mathrm{i}}\,\dot{a}}\frac{\partial}{\partial y^{a}_{\bar{\mathrm{i}}}}\right) g^{(m,\overline{m})}\left(\lambda_{\mathrm{i}}, y_{\bar{\mathrm{i}}}; \widetilde{y}_{\mathrm{j}}, \widetilde{\lambda}_{\bar{\mathrm{j}}}\middle| x\right) = 0$$

with $\mathrm{i},\mathrm{j} = 1\ldots m$ and $\bar{\mathrm{i}},\bar{\mathrm{j}} = 1\ldots\overline{m}$. Characteristics

$$\lambda_{\mathrm{i}a}, \quad \widetilde{\lambda}_{\bar{\mathrm{i}}\dot{a}}, \quad y^{a}_{\bar{\mathrm{i}}} + x^{a\dot{a}}\,\widetilde{\lambda}_{\bar{\mathrm{i}}\dot{a}}, \quad \widetilde{y}^{\dot{a}}_{\mathrm{i}} + x^{a\dot{a}}\,\lambda_{\mathrm{i}a}$$

- Plane-wave solution

$$\chi_{\mu,\widetilde{\mu}} = \exp\left[i\left(\widetilde{\mu}^{\mathrm{i}}_{\dot{a}}\left(\widetilde{y}^{\dot{a}}_{\mathrm{i}} + x^{a\dot{a}}\lambda_{\mathrm{i}\,a}\right) + \mu^{\bar{\mathrm{j}}}_{a}\left(y^{a}_{\bar{\mathrm{j}}} + x^{a\dot{a}}\widetilde{\lambda}_{\bar{\mathrm{j}}\,\dot{a}}\right)\right)\right]\cdot$$
$$\prod_{\mathrm{j},\,a}\delta\left(\lambda^{\mathrm{j}}_{a} - \mu^{\mathrm{j}}_{a}\right)\prod_{\bar{\mathrm{j}},\,\dot{a}}\delta\left(\widetilde{\lambda}^{\bar{\mathrm{j}}}_{\dot{a}} - \widetilde{\mu}^{\bar{\mathrm{j}}}_{\dot{a}}\right)$$

- Useful x-independent solution $\rho_{\mathrm{i}\bar{\mathrm{j}}} = \lambda_{\mathrm{i}\,a} y^{a}_{\bar{\mathrm{j}}} - \widetilde{\lambda}_{\bar{\mathrm{j}}\,\dot{a}}\widetilde{y}^{\dot{a}}_{\mathrm{i}}$ (Arkani-Hamed et al.)

Conserved charges

- Turning unfolded equations into the system of first-order PDE's gives a number of useful options:
  1. On-shell-closed differential form
     $$\Omega^{(m,\overline{m})}[g] = \mathrm{d}^{Km}\lambda\,\mathrm{d}^{K\overline{m}}\widetilde{\lambda}\,\mathrm{d}^{K\overline{m}}\big(y + x\widetilde{\lambda}\big)\,\mathrm{d}^{Km}\big(\widetilde{y} + x\lambda\big)\;g^{(m,\overline{m})}$$
  2. For a given solution $g^{(m,\overline{m})}$ there is functional ambiguity $g^{(m,\overline{m})} \to \eta\, g^{(m,\overline{m})}$, where "parameter" η is a solution of rank-**r** equation.
- Integration
  $$\mathcal{Q}^{(m,\overline{m})} = \int_{\Sigma} \Omega^{(m,\overline{m})}$$
  is independent of local variations of Σ, giving multiparticle conserved charge.

Reproducing MHV 3-point amplitude

$$\mathcal{Q}^{(m,\overline{m})} = \int_\Sigma \mathrm{d}^{Km}\lambda\,\mathrm{d}^{K\overline{m}}\widetilde{\lambda}\,\mathrm{d}^{K\overline{m}}\big(y + x\widetilde{\lambda}\big)\,\mathrm{d}^{Km}\big(\widetilde{y} + x\lambda\big)\ g^{(m,\overline{m})}$$

- $\mathbf{r} = 3$, consider $m = 2, \overline{m} = 1$: $\mathrm{j} = 1, 2$, $\bar{\mathrm{j}} = 3$
- Σ such that $x = 0$
- $g^{(2,1)} = \mathrm{sign}\,(\rho_{13})\,\mathrm{sign}\,(\rho_{23})\ \chi_{\mu,\widetilde{\mu}}$

$$\mathcal{Q}^{(2,1)} = \int \mathrm{d}^2 y\,\mathrm{d}^4\widetilde{y}\,\exp\left(i\widetilde{\mu}_1\widetilde{y}_1 + i\widetilde{\mu}_2\widetilde{y}_2 + i\mu_3 y_3\right)$$

$$\mathrm{sign}\big(\mu_1 y_3 - \widetilde{\mu}_3\widetilde{y}_1\big)\,\mathrm{sign}\big(\mu_2 y_3 - \widetilde{\mu}_3\widetilde{y}_2\big).$$

- Change of 6 variables from $y_{\bar{\mathrm{i}}}^a$, $\widetilde{y}_{\mathrm{j}}^a$ to 2 variables ρ_{13}, ρ_{23} and 4 variables $\widetilde{\xi}_i^{\dot{a}}$ parametrizing the kernel of

$$\rho_{\mathrm{i}\bar{\mathrm{j}}} = \mu_{\mathrm{i}\,a} y_{\bar{\mathrm{j}}}^a - \widetilde{\mu}_{\bar{\mathrm{j}}\,\dot{a}}\widetilde{y}_{\mathrm{i}}^{\dot{a}}$$

Reproducing MHV 3-point amplitude

$$\mathcal{Q}^{(2,1)} = \int \mathrm{d}^2 y \, \mathrm{d}^4 \widetilde{y} \, \exp\left(i\widetilde{\mu}_1 \widetilde{y}_1 + i\widetilde{\mu}_2 \widetilde{y}_2 + i\mu_3 y_3\right)$$
$$\mathrm{sign}\big(\mu_1 y_3 - \widetilde{\mu}_3 \widetilde{y}_1\big) \, \mathrm{sign}\big(\mu_2 y_3 - \widetilde{\mu}_3 \widetilde{y}_2\big).$$

- Change of variables gives the following Jacobian

$$\mathrm{d}^2 y \, \mathrm{d}^4 \widetilde{y} = \langle \mu_1 \mu_2 \rangle^5 \, \mathrm{d}^2 \rho \, \mathrm{d}^4 \widetilde{\xi}$$

- Exponent transforms into

$$\exp\left[i \, \rho_{\mathrm{i}\bar{\mathrm{j}}} \, \langle \mu_{\bar{\mathrm{j}}} \, \varepsilon_{\mathrm{ik}} \mu_{\mathrm{k}} \rangle\right] \, \exp\left[i \, \widetilde{\xi}_{\mathrm{i}}^{\dot{a}} F_{\mathrm{i}\,\dot{a}}\right],$$

where $F_{i\dot{a}} = \langle \mu_1 \mu_2 \rangle \, \widetilde{\mu}_{\mathrm{i}\dot{a}} + \langle \mu_{\bar{\mathrm{j}}} \, \varepsilon_{\mathrm{ik}} \mu_{\mathrm{k}} \rangle \, \widetilde{\mu}_{\bar{\mathrm{j}}\dot{a}}$

- We make use of the integrals

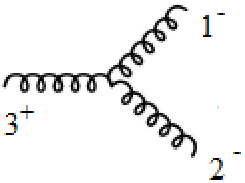
$$\int_{-\infty}^{+\infty} \mathrm{d}x \, e^{ikx} \, \mathrm{sign}\,(x) = \frac{2i}{k} \quad \text{and} \quad \int_{-\infty}^{+\infty} \mathrm{d}x \, e^{ikx} = 2\pi \, \delta(k)$$

Reproducing MHV 3-point amplitude

We arrive at the desired result

$$\mathcal{Q}^{(2,1)} \propto \frac{\langle \mu_1 \mu_2 \rangle^3}{\langle \mu_2 \mu_3 \rangle \langle \mu_3 \mu_1 \rangle} \, \delta^4 \Big(\sum_I \mu_{Ia} \widetilde{\mu}_{I\dot{a}} \Big),$$

of the MHV 3-point function



Reproducing MHV **r**-point amplitude

- $m = 2$, $\overline{m} = \mathbf{r} - 2$
- Σ such that $x = 0$
- $g^{(2,\mathbf{r}-2)} = \prod_{\mathrm{i},\bar{\mathrm{j}}} \mathrm{sign}(\rho_{\mathrm{i}\bar{\mathrm{j}}})\, \chi_{\mu,\widetilde{\mu}}$
- ...

However

$$\mathcal{Q}^{(2,\mathbf{r}-2)} \propto \frac{\langle \mu_1 \mu_2 \rangle^{\mathbf{r}}}{\prod_{\mathrm{i}=1,2,\bar{\mathrm{j}}=\bar{3}\ldots\bar{n}} \langle \mu_{\mathrm{i}} \mu_{\bar{\mathrm{j}}} \rangle}\, \delta^4\Big(\sum_I \mu_{Ia} \widetilde{\mu}_{I\dot{a}} \Big)$$

- We still have ambiguity in the choice of "parameter" η: rational function

$$\eta_\mu^{\mathrm{extra}} = \frac{\langle \lambda_1\, \mu_{\bar{3}} \rangle \ldots \langle \lambda_1\, \mu_{\bar{\mathbf{r}}} \rangle \cdot \langle \lambda_2\, \mu_{\bar{3}} \rangle \ldots \langle \lambda_2\, \mu_{\bar{\mathbf{r}}} \rangle}{\langle \lambda_1\, \lambda_2 \rangle^{\mathbf{r}-3} \langle \lambda_2\, \mu_{\bar{3}} \rangle \langle \mu_{\bar{4}}\, \mu_{\bar{5}} \rangle \ldots \langle \mu_{\overline{\mathbf{r}-1}}\, \mu_{\bar{\mathbf{r}}} \rangle \langle \mu_{\bar{\mathbf{r}}}\, \lambda_1 \rangle}$$

brings $\mathcal{Q}^{(2,\mathbf{r}-2)}$ to MHV **r**-point amplitude.

Conclusion

- Scattering amplitudes and multiparticle HS charges are indeed related: examples of reproducing MHV amplitudes
- There is a vast ambiguity in the construction of Ω, giving possibility to reproduce amplitudes by adjusting "parameter" η
- Spin information has to be specified within the formalism
- Particular form of amplitudes may be fixed only by considering non-linear equations