

# Conformal Higher Spin action from worldline path integrals

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## Motivations

- Conformal Higher Spin (CHS) theories generalize four dimensional Maxwell and Weyl squared gravity
- In the supersymmetric case, they are HS counterpart of conformal supergravities
- Remarkably, admit a consistent non-linear action around flat space
- Contrary to massless HS in AdS, conformal invariance fixes the derivative structure of vertices  $\longrightarrow$  finite derivatives for given spins and number of fields
- Related to boundary values of massless HS in AdS via AdS/CFT
- Despite being non-unitary, there are hints for unitary truncations at tree-level

## Outline

- Free theory
- Low spin case: conformal (super)gravity as an induced action
- HS conformal currents from free fields
- CHS action as an induced action
- Worldline model of scalar in CHS background
- Worldline path integral representation for all vertices
- Conclusions

## Free theory

- Consider four dimensional Maxwell + Weyl squared gravity

$$S[A, g] = \int d^4x \sqrt{-g} [F^{\mu\nu} F_{\mu\nu} + C^{\mu\nu\lambda\sigma} C_{\mu\nu\lambda\sigma}]$$

invariant under  $U(1)$ , diffeos and Weyl local symmetries

- Linearize around flat spacetime  $\rightarrow$

$$S_2[A, h] = \int d^4x [A^\mu P_\mu{}^\nu \square A_\nu + h^{\mu\nu} P_{\mu\nu}{}^{\lambda\sigma} \square^2 h_{\lambda\sigma}]$$

where  $P_1$  and  $P_2$  are transverse-traceless projectors of spin one and two

- Invariant under linearized gauge symmetry

$$\delta A_\mu = \partial_\mu \epsilon, \quad \delta h_{\mu\nu} = \partial_{(\mu} \epsilon_{\nu)} + \alpha \eta_{\mu\nu}$$

## Free theory

- Natural generalization to spin  $s$

Fradkin, Tseytlin (1989)

$$S[h] = \int d^4x \sum_{s=0}^{\infty} h^{\mu(s)} P_{\mu(s)}^{\nu(s)} \square^s h_{\nu(s)}$$

where  $P_s$  are spin- $s$  transverse-traceless projectors

- Invariant under maximal spin- $s$  gauge symmetry compatible with locality

$$\delta h_{\mu(s)} = \partial_{\mu} \epsilon_{\mu(s-1)} + \eta_{\mu\mu} \alpha_{\mu(s-2)}$$

- In higher even  $d$ ,  $\square^s \rightarrow \square^{s+\frac{d-4}{2}}$  with conformal dimension  $\Delta_s = 2 - s$
- In general non-unitary for  $s \geq 2$
- Although there are conformal scalars with two or higher derivatives, this  $h_0$  is auxiliary in  $d = 4$

## Induced conformal gravity

- How do we construct the interacting theory? Noether method? Maybe
- Suppose one doesn't know Maxwell and Weyl squared gravity, and take a scalar conformally coupled to the metric +  $U(1)$  coupling

$$S[\phi; A, g] = \int d^4x \sqrt{g} \left[ g^{\mu\nu} D_\mu \phi^* D_\nu \phi + \frac{1}{6} R \phi^* \phi \right], \quad D_\mu = \partial_\mu - iA_\mu$$

- Compute the scalar effective action

$$\Gamma[A, g] = a_0 \Lambda^4 \mathcal{V} + a_2 \log \Lambda \int d^4x \sqrt{g} [F_{\mu\nu}^2 + C_{\mu\nu\lambda\sigma}^2] + \text{finite}$$

and "discover" Maxwell+Weyl squared from the log divergent piece

- Notice that around flat space

$$e^{-\Gamma[A, h]} = \left\langle e^{\int J^\mu A_\mu + T_{\text{conf}}^{\mu\nu} h_{\mu\nu} + \mathcal{O}(A, h)^2} \right\rangle_{\text{CFT}}$$

$\mathcal{N} = 4$  SYM, AdS/CFT and  $\mathcal{N} = 4$  C-sugra Liu, Tseytlin (1998)

- Similar construction starting from  $\mathcal{N} = 4$  SYM: it can be naturally coupled to  $\mathcal{N} = 4$  conformal supergravity

Bergshoeff, de Roo, de Wit (1981); de Roo, Wagemans (1985)

- The SYM partition function in C-sugra background

$$e^{-\Gamma[\mathcal{G}]} = \int \mathcal{D}\mathcal{A} e^{-S_{\text{SYM}}[\mathcal{A}; \mathcal{G}]}$$

yields the C-sugra action as log divergent part

$$\Gamma[\mathcal{G}] = \Gamma_{\text{fin}}[\mathcal{G}] + a_2 \log \Lambda S_{\text{CSG}}[\mathcal{G}]$$

- Again, the part of  $S_{\text{SYM}}[\mathcal{A}; \mathcal{G}]$  linear in CSG fluctuations yields the Noether couplings to SYM operators, so that

$$e^{-\Gamma[\mathcal{G}]} = \left\langle e^{\int O(\mathcal{A})_a \mathcal{G}^a + \mathcal{O}(\mathcal{G}^2)} \right\rangle_{\text{SYM}}$$

that corresponds, at large 't Hooft coupling, to on-shell  $\mathcal{N} = 8$  gauged sugra in  $AdS_5$

## Induced CHS action

- Apply the same idea to the simplest CFT of free scalar field in even  $d$

$$S_0[\phi] = \int d^d x \partial^\mu \phi^* \partial_\mu \phi$$

- It possesses infinitely many traceless conserved currents  $J_s = \phi^* \partial^s \phi + \dots$  of conformal dimension  $\Delta_{J_s} = d - 2 + s$
- Low spin are scalar operator  $J_0 = \phi^* \phi$ ,  $U(1)$  current  $J_\mu = \phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*$  and traceless stress energy tensor  $J_{\mu\nu} = T_{\mu\nu}^{\text{conf}} = \partial_{(\mu} \phi^* \partial_{\nu)} \phi + \dots$
- Couple to sources  
 $S[\phi; h] = S_0[\phi] + \sum_{s=0}^{\infty} \frac{(i)^s}{s!} \int d^d x J^{\mu(s)} h_{\mu(s)}$
- Sources  $h_s$  have dimension  $\Delta_{h_s} = 2 - s$  and inherit linearized invariance on the scalar mass-shell under  $\delta h_s = \partial \epsilon_{s-1} + \eta \alpha_{s-2} \rightarrow$  **identified as CHS**



## Induced CHS action

- Consider the scalar path integral

$$\int \mathcal{D}\phi^* \mathcal{D}\phi e^{-S[\phi;h]} := e^{-\Gamma[h]} = \left\langle e^{\int \sum_s J_s \cdot h_s} \right\rangle_{\text{free CFT}}$$

- According to HS AdS/CFT [Sezgin, Sundell; Klebanov, Polyakov \(2002\)](#) conformal spin- $s$  conserved currents are dual to AdS massless spin- $s$  gauge fields  $\longrightarrow$
- the above should correspond to the on-shell AdS action of interacting MHS
- From previous discussion, log divergent part of  $\Gamma[h]$  defines non-linear CHS action [Tseytlin; Segal \(2002\); Bekaert, Joung, Mourad \(2010\)](#)  
$$\Gamma[h] = \sum_{n=1}^{\infty} \Lambda^{2n} \Gamma_n[h] + \log \Lambda \mathcal{S}_{\text{CHS}}[h] + \Gamma_{\text{fin}}[h]$$

## Currents, symmetries, effective action

- In terms of the CHS generating function

$h(x, u) := \sum_{s=0}^{\infty} \frac{1}{s!} h_{\mu(s)} u^{\mu_1} \dots u^{\mu_s}$  the scalar action can be recast as (basically IBP derivatives on  $\phi^*$ )

$$S_0[\phi] + \int d^d x J(x, i\partial_u) h(x, u)|_{u=0} = \langle \phi | \hat{P}^2 + \hat{H}(\hat{X}, \hat{P}) | \phi \rangle$$

where  $\hat{X}^\mu \equiv x^\mu$  and  $\hat{P}_\mu \equiv -i\partial_\mu$

- Invariant under the gauge symmetry

$$|\phi\rangle \rightarrow \hat{O}^{-1} |\phi\rangle \quad (\hat{P}^2 + \hat{H}) \rightarrow \hat{O}^\dagger (\hat{P}^2 + \hat{H}) \hat{O}$$

- For  $\hat{O}(\hat{X}, \hat{P}) = e^{\hat{A} + i\hat{E}}$  at lowest order  $\hat{E}$  generates the  $\epsilon$  differential symmetries, and  $\hat{A}$  the HS Weyl symmetries  $\alpha$
- The effective action is thus given by the functional determinant  $\Gamma[h] = \text{Tr} \log(\hat{P}^2 + \hat{H})$
- Next step: worldline representation of  $\Gamma[h]$

## Worldline formalism

- The WL formalism is quite effective for such one-loop effective actions. E.G. scalar particle in gravitational and e.m. background

$$S_{g,A}[x, p, e] = \int_0^1 d\tau \left\{ p_\mu \dot{x}^\mu - \frac{e}{2} [g^{\mu\nu} (p_\mu - A_\mu)(p_\nu - A_\nu) + \xi R(g)] \right\}$$

or

$$S_{g,A}[x, e] = \int_0^1 d\tau \left[ \frac{1}{2e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + A_\mu \dot{x}^\mu + \xi \frac{e}{2} R \right]$$

- $e(\tau)$  is the einbein, gauges relativistic hamiltonian and ensures  $\tau$ -rep invariance
- Path integral on the circle produces the QFT one-loop photon-graviton EA

$$\Gamma[A, g] = \int_{S^1} \frac{DxDe}{\text{VolG}} e^{-S_{g,A}[x,e]}$$

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$$\Gamma[A, g] = \int_{S^1} \frac{DxDe}{\text{VolG}} e^{-S_{g,A}[X, e]}$$

- $e(\tau)$  has a zero mode on  $S^1 \rightarrow$  gauge fixing  $e(\tau) = T$  leaves a modular integration. FP action is locally trivial, but yields  $T^{-1}$  factor on  $S^1$  topology

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- Gauge fixed path integral yields heat kernel expansion

$$\Gamma[h] = \int_0^\infty \frac{dT}{T} \int_{S^1} Dx e^{-S_{g,A}[x,T]} = \int_0^\infty \frac{dT}{T} K[T; g, A]$$

where  $[T] = M^{-2}$  is the proper time and  $T \rightarrow 0$  is UV region of the QFT (1d version of torus modular parameter  $|\tau|$  in String Theory)

## Scalar particle in CHS background

- Generalize the hamiltonian action de Wit, Freedman (1980); Segal (2002)

$$S_h[x, p, e] = \int_0^1 d\tau \left[ p_\mu \dot{x}^\mu - e G(x, p) \right]$$

$$G(x, p) = p^2 + \mathcal{H}(x, p)$$

- The CHS basis is  $h(x, p)$  with

$$\mathcal{H}(x, p) = \mathcal{P}_d h(x, p) = \sum_{s=0}^{\infty} \sum_{n=0}^{[s/2]} c_n(s, d) \left[ \partial^{*2} - \text{Tr}\square \right]^n h_s(x, p)$$

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$$G(x, p) = p^2 + \mathcal{H}(x, p)$$

- Relativistic invariance under  $\tau$ -rep with transformations

$$\delta x^\mu = \xi \{x^\mu, G\}_{\text{P.B.}}, \quad \delta p_\mu = \xi \{p_\mu, G\}_{\text{P.B.}}, \quad \delta e = \dot{\xi}$$

## Scalar particle in CHS background

- Generalize the hamiltonian action [de Wit, Freedman \(1980\)](#); [Segal \(2002\)](#)

$$S_h[x, p, e] = \int_0^1 d\tau \left[ p_\mu \dot{x}^\mu - e G(x, p) \right]$$
$$G(x, p) = p^2 + \mathcal{H}(x, p)$$

- CHS background gauge symmetry under

$$\delta \mathcal{H}(x, p) = \{ \epsilon(x, p), p^2 + \mathcal{H}(x, p) \}_{\text{P.B.}} + \alpha(x, p) (p^2 + \mathcal{H}(x, p))$$

with WL variables transforming as

$$\delta x^\mu = \{ x^\mu, \epsilon(x, p) \}_{\text{P.B.}}, \quad \delta p_\mu = \{ p_\mu, \epsilon(x, p) \}_{\text{P.B.}}, \quad \delta e = -\alpha(x, p) e$$

- Notice that the  $\alpha$ -symmetry is broken by gauge fixing  
 $e(\tau) = T$



## Caveat: covariant vs Noether coupling

As in the field theoretic description, the low spin fields do not coincide with the covariant ones (besides the map  $\mathcal{H} = \mathcal{P}_d h$ )

- Take the covariant hamiltonian

$$G_{\text{cov}}(x, p) = g^{\mu\nu} (p_\mu - A_\mu)(p_\nu - A_\nu) + \xi R(g), \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}^{\text{w}}$$

- Compare with low spin branch of  $\mathcal{H}$

$$G_{\text{Noether}}(x, p) = p^2 + h_0 - 2 h^\mu p_\mu - h^{\mu\nu} p_\mu p_\nu$$

- One has the relation

$$h_{\mu\nu} = h_{\mu\nu}^{\text{w}} + r_{\mu\nu}(h_{\text{w}}) \quad (g^{\mu\nu} = \eta^{\mu\nu} - h_{\text{w}}^{\mu\nu} - r^{\mu\nu}(h_{\text{w}}))$$

$$h_\mu = A_\mu - h_{\mu\nu}^{\text{w}} A^\nu - r_{\mu\nu}(h_{\text{w}}) A^\nu$$

$$h_0 = R(\eta + h_{\text{w}}) + A^2 - A_\mu A_\nu (h_{\text{w}}^{\mu\nu} + r^{\mu\nu}(h_{\text{w}}))$$

- Compositeness of  $h_0$  can be deduced, for instance, by its  $U(1)$  transformation  $\delta_{U(1)} h_0 = 2g^{\mu\nu} A_\mu \partial_\nu \epsilon$

## CHS action from WL path integral

- CHS effective action from WL path integral on the circle

$$\Gamma[h] = \int_{S^1} \frac{Dx Dp D e}{\text{Vol G}} e^{-S_h[x,p,e]}$$

- Gauge fixed path integral  $e(\tau) = T \rightarrow$  Schwinger proper time expansion

$$\Gamma_{\Lambda}[h] = \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{dT}{T} K[T; h]$$

- Log divergent piece is the  $\mathcal{O}(T^0)$  part of  $K[T; h]$

## CHS action from WL path integral

- Gauge fixed path integral  $\Gamma_{\wedge}[h] = \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{dT}{T} K[T; h]$
- Trace of heat kernel  $K[T; h] = \int_{S^1} Dx Dp e^{-S_h[x, p; T]}$
- Extract  $x$  zero mode:  $x(\tau) = x + q(\tau)$  and average w.r.t. free action

$$K[T; h] = \int \frac{d^d x}{(4\pi T)^{d/2}} \left\langle e^{-T \int_0^1 d\tau \mathcal{H}(x+q, p)} \right\rangle = \int \frac{d^d x}{(4\pi T)^{d/2}} \sum_{n=0}^{\infty} T^n \mathcal{V}_n[T; h]$$

- $(4\pi T)^{-d/2}$  is the free PI on the circle;  $\mathcal{V}_n[T; h]$  yields the  $n$ -point effective vertex.  $\mathcal{V}_n$  is analytic in  $T$  for  $s \leq 2$ , but *not* for HS

## CHS action from WL path integral

- Gauge fixed path integral  $\Gamma_{\Lambda}[h] = \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{dT}{T} K[T; h]$

$$K[T; h] = \int \frac{d^d x}{(4\pi T)^{d/2}} \sum_{n=0}^{\infty} T^n \mathcal{V}_n[T; h]$$

- Effective vertex is given by a differential operator acting on a string of  $\mathcal{H}$ 's

$$\mathcal{V}_n[T; h] = \hat{V}_n(T; \partial_{x_i}, \partial_{u_i}) \mathcal{H}(x_1, u_1) \cdots \mathcal{H}(x_n, u_n) \Big|_{\substack{x_i=x \\ u_i=0}}$$

(expand  $\mathcal{H}(x + q_i, p_i)$  around  $(x, 0)$  and compute exact PI)

- Freedom in IBP expressed as  $\sum_{i=1}^n \partial_{x_i} \sim 0$

## CHS action from WL path integral

- Gauge fixed path integral  $\Gamma_{\Lambda}[h] = \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{dT}{T} K[T; h]$

$$K[T; h] = \int \frac{d^d x}{(4\pi T)^{d/2}} \sum_{n=0}^{\infty} T^n \mathcal{V}_n[T; h]$$

- $\mathcal{V}_n[T; h] = \hat{V}_n(T; \partial_{x_i}, \partial_{u_i}) \mathcal{H}(x_1, u_1) \cdots \mathcal{H}(x_n, u_n) \Big|_{\substack{x_i=x \\ u_i=0}}$
- WL path integral yields the result in terms of parametric proper time integrals

$$\hat{V}_n(T; \partial_{x_i}, \partial_{u_i}) = \frac{(-1)^n}{n} e^{\frac{1}{4T} \partial_U^2} \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-2}} d\tau_{n-1}$$

$$\times \exp \sum_{\substack{(\tau_n=0) \\ i < j}} \left\{ -i(\tau_{ij} - \frac{1}{2})(\partial_{x_i} \cdot \partial_{u_j} - \partial_{x_j} \cdot \partial_{u_i}) + T \tau_{ij}(\tau_{ij} - 1) \partial_{x_i} \cdot \partial_{x_j} \right\}$$

## CHS action from WL path integral

- $\mathcal{V}_n[T; h] = \hat{V}_n(T; \partial_{x_i}, \partial_{u_i}) \mathcal{H}(x_1, u_1) \cdots \mathcal{H}(x_n, u_n) \Big|_{\substack{x_i=x \\ u_i=0}}$

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- When acting on a definite spin basis  $\{s_i\}$  of  $h(x_i, u_i)$  there are at most  $s_{\text{tot}} = \sum_{i=1}^n s_i$   $u$ -derivatives  $\rightarrow$  only polynomials from the inverse  $T$  and  $T$ -independent exp

## CHS action from WL path integral

- $\mathcal{V}_n[T; h] = \hat{V}_n(T; \partial_{x_i}, \partial_{u_i}) \mathcal{H}(x_1, u_1) \cdots \mathcal{H}(x_n, u_n) \Big|_{\substack{x_i=x \\ u_i=0}}$

$$\hat{V}_n(T; \partial_{x_i}, \partial_{u_i}) = \frac{(-1)^n}{n} e^{\frac{1}{4T} \partial_U^2} \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-2}} d\tau_{n-1}$$
$$\times \exp \sum_{\substack{(\tau_n=0) \\ i < j}} \left\{ -i(\tau_{ij} - \frac{1}{2})(\partial_{x_i} \cdot \partial_{u_j} - \partial_{x_j} \cdot \partial_{u_i}) + T \tau_{ij}(\tau_{ij} - 1) \partial_{x_i} \cdot \partial_{x_j} \right\}$$

- For every  $s_{\text{tot}}$ ,  $\hat{V}_n$  has a well defined Laurent expansion in  $T$

$$\mathcal{V}_n[T; h] = \sum_{k=-\infty}^{\infty} T^k \mathcal{V}_n^{(k)}[h]$$

and the component giving rise to the log is  $k = \frac{d}{2} - n$

## CHS action from WL path integral

Finally, the classical CHS action is given by

$$S_{\text{CHS}}[h] = \int d^d x \sum_{n=2}^{\infty} \mathcal{V}_n^{(d/2-n)}[h]$$

- For a given set of spins  $\{s_j\}$  with  $\sum_{i=1}^n s_i = s_{\text{tot}}$  locality is manifest at all orders  $n$ . In fact, the number of derivatives  $p$  is fixed to

$$p = d + s_{\text{tot}} - 2n$$

- E.g. the diagonal quadratic part has  $p = d - 4 + 2s$
- Only spin one in 4d:  $p = 4 - n$  accounts for YM cubic and quartic vertices
- Only spin two in 4d:  $p = 4$



## CHS action from WL path integral

Finally, the classical CHS action is given by

$$S_{\text{CHS}}[h] = \int d^d x \sum_{n=2}^{\infty} \nu_n^{(d/2-n)}[h]$$

- The quadratic part has been rederived and indeed reproduces Fradkin-Tseytlin

$$S_2[h] = \sum_{s=0}^{\infty} c_s \int d^d x [h_s P_s \square^{s+\frac{d-4}{2}} h_s]$$

- Mind that the differential operator  $\hat{V}_n^{(d/2-n)}$  has to undo the map  $\mathcal{H} = \mathcal{P}_d h$
- The constants  $c_s$  are irrelevant at the free level, but they set the relative normalization of all couplings

## Conclusions and outlook

- WL representation of CHS vertices, similar to string 1st quantized approach
- Technically annoying fact: the map  $\mathcal{H} = \mathcal{P}_d h \rightarrow$  In TT gauge reduces to identity and, in 4d, one could use spinors basis to simplify
- Zero total dof, anomaly cancellations and hints for S-matrix triviality [Beccaria, Bekaert, Joung, Mourad, Tseytlin; McLoughlin, Hähnel \(2013→ now\)](#) rely on regulated spin sums  $\rightarrow$  develop a 1st quantized WL or worldsheet model for CHS, would help understanding
- WL approach can simplify manifest covariant coupling to spin 2 [Beccaria, Grigoriev, Tseytlin \(2016-2017\)](#)
- Possible unitary truncations of amplitudes?  
[McLoughlin, Hähnel \(2016-2017\)](#)

THANKS FOR YOU ATTENTION!