Continuous spin fields of mixed-symmetry type

Konstantin Alkalaev

Tamm Theory Department, Lebedev Physical Institute

K.Alkalaev, M. Grigoriev, arXiv:1712.02317

HSTH-7, Moscow 2018

Motivation

- Continuous spin fields are massless $m = 0$; the dimensionful parameter μ (Bargmann, Wigner 1948); infinite number of PDoF.
- Continuous spin dynamics can be defined on the space of fields which is the sum of Fronsdal-like rank-s fields with $s = 0, ..., \infty$, similar to the standard *interacting* HS theories (Fradkin, Vasiliev 1986).
- Action functional on Minkowski space and AdS is the infinite sum of Fronsdal rank-s actions with off-diagonal terms proportional to μ (Schuster, Toro 2014, Metsaev 2016). The gauge transformations are the standard Fronsdal transformations deformed by Stueckelberg-like terms also proportional to μ.

Outline

- Group-theoretical description
- Howe duality and higher spin fields
- Equations of motion as constraints
- Triplet formulation
- Metric-like fields and the Schuster-Toro representation
- Light-cone formulation
- Concluding remarks

Group-theoretical description

Continuous spin particles correspond to infinite-dimensional massless UIRs of the Poincare algebra $iso(d-1,1)$, induced from *infinite-dimensional* UIRs of $iso(d-2)$ subalgebra. Bargmann, Wigner 1948.

Quantum numbers

- a mass $m = 0$
- a continuous spin parameter $\mu \neq 0$
- (half-)integer spin wights $(s_1, ..., s_p)$, where $p = [\frac{d-3}{2}]$.

Casimir operators Generalized Pauli-Lubanski tensors

$$W_{m_1 \dots m_k} = \epsilon_{m_1 \dots m_k a_{k+1} \dots a_d} P^{a_{k+1}} M^{a_{k+2} a_{k+3}} \dots M^{a_{d-1} a_d}$$

The Pauli-Lubanski tensors covariantly transform under Lorentz subalgebra $o(d-1,1)$ and satisfy $[P_a, W_{m_1 \dots m_k}] = 0$ so that the Casimir operators can be given as

$$C_{2p} = W_{m_1 \dots m_{p-1}} W^{m_1 \dots m_{p-1}}$$

For arbitrary representations the Casimir operators can be rather complicated, but in the massless case $C_2 \equiv P^2 = 0$ they are drastically simplified. Denoting $\pi_a = M_{ab} P^b$ we find the general expression

$$C_{2p} \approx \left[a_{p,0} + a_{p,2}\, M^2 + ... + a_{p,2p-4}\, M^{2p-4}\right] \pi_a \pi^a$$

E.g., the quartic Casimir operator is given by $C_4 \sim \pi_a \pi^a$. Then,

- $C_2 = 0$ defines a masslessness
- C_4 yields a continuous spin value μ^2 (Brink et al 2002)
- $C_6, C_8, ...$ yield spin weights

In other words, a continuous spin representation is characterized by the parameter μ and $s_1, ..., s_p$. The short little algebra $o(d-3)$.

Generating function in auxiliary variables

Two types of indices A_I^a running $a = 0, ..., d-1$ and $I = 0, ..., n-1$. We consider polynomials

$$\phi(A) = \sum \phi_{a_1\ \dots\ a_{m_0};\ \dots\dots\ ;c_1\ \dots\ c_{m_{n-1}}} A_0^{a_1} \cdots A_0^{a_{m_0}}\ \cdots\ A_{n-1}^{c_1} \cdots A_{n-1}^{c_{m_{n-1}}}$$

Orthogonal algebra $o(d-1,1)$:

$$J^{ab} = A_I^a \frac{\partial}{\partial A_{bI}} - A_I^b \frac{\partial}{\partial A_{aI}}$$

rotations

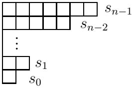
Symplectic algebra $sp(2n)$:

$$T_{IJ} = A_I^a A_{aJ}\,, \qquad T_I{}^J = \frac{1}{2}\,\{A_I^a, \frac{\partial}{\partial A_J^a}\}\,, \qquad T^{IJ} = \frac{\partial}{\partial A_I^a}\frac{\partial}{\partial A_{aJ}}$$

trace creation — *Young symmetrizer* — *trace annihilation*

Howe duality

Finite-dimensional irrep of $o(d-1,1)$ algebra



$\Uparrow$

Howe duality

$\Downarrow$

Highest weight conditions of $sp(2n)$ algebra

$$T_I{}^I \phi = s_I \, \phi$$

$$T^{IJ}\phi = 0\,, \qquad\qquad T_I{}^J\phi = 0 \quad I > J$$

Introducing Poincare algebra

Remarkably, the auxiliary variables allow us to realize the Poincare algebra as well. Manifest $sp(2n+2)$ is broken to $sp(2n)$

$$A_0^a \equiv x^a \ , \qquad A_I^a \equiv a_i^a \ , \quad i = 1, ..., n$$

The Poincare algebra $iso(d-1,1)$ basis elements are realized as

$$P_a = \frac{\partial}{\partial x^a} \ , \qquad M_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} + a_{a\,i} \frac{\partial}{\partial a_i^b} - a_{b\,i} \frac{\partial}{\partial a_i^a} \ .$$

Let us introduce notation

$$\Box = \frac{\partial^2}{\partial x^b \partial x_b} \ , \qquad D_i^\dagger = a_i^b \frac{\partial}{\partial x^b} \ , \qquad D^i = \frac{\partial^2}{\partial a_i^b \partial x_b} \ ,$$

$$T^{ij} = \frac{\partial^2}{\partial a_{i\,b} \partial a_j^b} \ , \qquad T_{ij}^\dagger = a_i^b a_{bj} \ , \qquad N_i{}^j = a_i^b \frac{\partial}{\partial a_j^b} \ , \qquad N_i = a_i^b \frac{\partial}{\partial a_i^b} \ .$$

The above operators form a subalgebra in $sp(2n+2)$ algebra dual to the Lorentz algebra $o(d-1,1)$. The space of formal series in (x^b, a_i^b) is $iso(d-1,1) \oplus sp(2n+2)$ bimodule.

Equations of motion as constraints

Differential constraints

$$\Box\phi = 0\,, \qquad D^i\phi = 0\,, \qquad i = 1, ..., n\,.$$

Algebraic constraints

$$(T^{ij} + \nu^{ij})\phi = 0\,, \qquad \nu^{ij} = \nu\,\delta^{1i}\delta^{1j}\,, \quad \nu \in \mathbb{R} \qquad i, j = 1, \ldots, n\,,$$

$$N_i{}^j\phi = 0 \quad i < j\,, \qquad N_i\phi = s_i\phi\,, \qquad i, j = 2, \ldots, n$$

Gauge equivalence. The gauge transformations are given by

$$\delta\phi = \left(D_i^\dagger + \mu_i\right)\chi^i\,, \qquad \mu_i = \mu\,\delta_{1i}\,, \quad \mu \in \mathbb{R} \qquad i = 1, \ldots, n$$

Comments:

- At $\mu, \nu = 0$ we reproduce the helicity case system (Alkalaev, Grigoriev, Tipunin 2008)
- The constraints are not the highest weight conditions of $sp(2n+2)$ algebra: they are typical for *the theory of coherent states*, where the states are defined as eigenstates of the annihilation operator. State do not diagonalize the spin weight operator N_1 anymore!
- A functional class: we take formal series in a_i^b satisfying the additional admissibility condition. A series f is admissible if its trace decomposition

$$f = f_0 + f_1^{ij}\,T_{ij}^\dagger + f_2^{ij,kl}\,T_{ij}^\dagger\,T_{kl}^\dagger + \ldots\,, \qquad T^{ij}f_p^{\cdots} = 0\,,$$

  is such that all coefficients are polynomials of finite order (i.e. for a given f there exists such $N \in \mathbb{N}$ that all f_r are of order not exceeding N).

Quadratic and quartic Casimir operator

Our formulation involves parameters μ, ν and $(n-1)$ spin weights $s_2, ..., s_n$. In d dimensions that allows describing all possible finite-dimensional modules of *the short little algebra* (Brink et al 2002)

$$o(d-3) \subset iso(d-2) \subset iso(d-1,1)$$

To characterize $iso(d-1,1)$ representations underlying our system we analyze the Casimir operators of the Poincare algebra.

- The quadratic Casimir operator $C_2 = P_a P^a \approx 0$ vanishes on-shell because of $\Box \approx 0$.
- The quartic Casimir operator $C_4 = (M_{ab}P^b)^2$ equals

$$C_4\, \phi(x,a) = -D_i^\dagger D_j^\dagger\, T^{ij}\, \phi(x,a) \approx\ \mu^2 \nu\, \phi(x,a)\ ,$$

  where we used the differential constraints, trace constrains along with the equivalence relation $\phi \sim \phi + (D^\dagger + \mu)\chi$ with the gauge parameter expressed in terms of the field ϕ.

Thus, the model propagates continuous spin particles, in which case fixing $\nu = 1$ we identify μ as the continuous spin parameter. Such a split between deformation parameters μ and ν is artificial and only their combination $\mu^2\nu$ has invariant meaning.

Triplet formulation

The triplet BRST operator is

$$\Omega = c_0\Box + c_i D^i + \left(D_i^\dagger + \mu_i\right)\frac{\partial}{\partial b_i} - c_i \frac{\partial}{\partial b_i}\frac{\partial}{\partial c_0}\,,$$

where $\mu_i = \mu\delta_{i1}$. It is defined on the subspace of $\Psi = \Psi(x, a|c, b)$ singled out by the BRST extended trace constraints

$$(\mathcal{T} + \nu)\Psi = 0\,, \qquad \mathcal{T}^{\alpha}\Psi = 0\,, \qquad \mathcal{T}^{\alpha\beta}\Psi = 0\,, \qquad \alpha, \beta = 2, ..., n$$

as well as the Young symmetry and the spin weight constraints

$$\mathcal{N}_\alpha{}^\beta\Psi = 0 \quad \alpha < \beta\,, \qquad \mathcal{N}_\alpha\Psi = s_\alpha\Psi\,.$$

The extended constraints read explicitly as

$$\mathcal{T}^{ij} = T^{ij} + \frac{\partial}{\partial c_i}\frac{\partial}{\partial b_j} + \frac{\partial}{\partial c_j}\frac{\partial}{\partial b_i}\,, \qquad \mathcal{N}_\alpha{}^\beta = N_\alpha{}^\beta + b_\alpha\frac{\partial}{\partial b_\beta} + c_\alpha\frac{\partial}{\partial c_\beta}\,,$$

$$\mathcal{N}_\alpha = N_\alpha + b_\alpha\frac{\partial}{\partial b_\alpha} + c_\alpha\frac{\partial}{\partial c_\alpha}\,, \qquad \alpha, \beta = 2, ..., n$$

Note that the triplet BRST operator is nilpotent $\Omega^2 = 0$ on the entire space of unconstrained fields and not only on the subspace singled out by the algebraic constraints.

Representing the ghost number-zero field $\Psi^{(0)}$ as $\Psi^{(0)} = \Phi + c_0 C$ we introduce component fields entering $\Phi = \Phi(x, a|b, c)$ and $C = C(x, a|b, c)$ according to

$$\Phi = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_k} \Phi^{i_1 \ldots i_k | j_1 \ldots j_k} , \qquad C = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_{k+1}} C^{i_1 \ldots i_k | j_1 \ldots j_{k+1}} .$$

These component fields can be identified as *generalized triplet fields* (Bengtsson 1986). The corresponding gauge transformation reads

$$\delta \Psi^{(0)} = \Omega \Psi^{(-1)} ,$$

where the ghost number -1 parameters $\Psi^{(-1)} = \Lambda + c_0 \Upsilon$ are given by

$$\Lambda = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_{k+1}} \Lambda^{i_1 \ldots i_k | j_1 \ldots j_{k+1}} , \qquad \Upsilon = \sum_k c_{i_1} \ldots c_{i_k} b_{j_1} \ldots b_{j_{k+2}} \Upsilon^{i_1 \ldots i_k | j_1 \ldots j_{k+2}} .$$

The triplet equations of motion for continuous spin fields have the form

$$\Omega \Psi^{(0)} = 0$$

Comments:

- The triplet BRST operator for the continuous spin system differs from the BRST operator for the helicity spin system by adding the term proportional to μ, i.e. $\Omega \to \Omega + \mu \frac{\partial}{\partial b}$.

Equivalent dynamical systems

Theory $(\mathcal{H}, \Omega)$:

- $\mathcal{H}$ – representation space of Ω, $\Omega^2 = 0$;
- Equations of motion $\Omega\phi = 0$, where $\phi \in \mathcal{H}$.

$$\text{Triplet } \mathcal{H} = \mathcal{E} \oplus \mathcal{F} \oplus \mathcal{G}$$

- $\mathcal{E}$ – dynamical fields
- $\mathcal{F}$ – auxiliary fields
- $\mathcal{G}$ – Stueckelberg fields

Theory $(\mathcal{E}, \hat{\Omega})$:

- $\mathcal{E}$ – representation space of $\hat{\Omega}$, $\hat{\Omega}^2 = 0$;
- Equations of motion $\hat{\Omega}\psi = 0$, where $\psi \in \mathcal{E}$.

$$(\mathcal{H}, \Omega) \text{ equivalent } (\mathcal{E}, \hat{\Omega})$$

Additional grading

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \qquad \Omega = \Omega_{-1} + \Omega_0 + \Omega_1 + \dots$$

Definition:

$$\mathcal{E} \oplus \mathcal{G} = Ker\Omega_{-1} \, , \qquad \mathcal{G} = Im\Omega_{-1} \, , \qquad \mathcal{E} = \frac{Ker\Omega_{-1}}{Im\Omega_{-1}}$$

See also:

- General approach (Barnich, Grigoriev, Semikhatov, Tipunin'04)
- Light cone DoF, quartets in string theory (Kato, Ogawa'83)
- Unfolded HS formulation (Lopatin, Vasiliev 1988, Shaynkman, Vasiliev'00)

Two reductions of the triplet formulation

- Metric-like formulation (deformed Fronsdal and Labastida formulations)

- Light-cone reduction

Metric-like formulation

Let the additional grading be a homogeneity degree in c_0. Then, the triplet BRST operator can be decomposed as $\Omega = \Omega_{-1} + \Omega_0 + \Omega_1$ with

$$\Omega_{-1} = -c_i \frac{\partial}{\partial b_i}\frac{\partial}{\partial c_0}\,, \qquad \Omega_0 = c_i D^i + \left(D_i^\dagger + \mu_i\right)\frac{\partial}{\partial b_i}\,, \qquad \Omega_1 = c_0 \Box\,.$$

The cohomology $H(\Omega_{-1})$ in ghost degree 0 and -1 can be explicitly described in terms of the lowest expansion components in ghosts c_i and b^i:

$$\Phi = \varphi + \dots$$
$$\Lambda^i = \chi^i + \dots$$

The lowest components φ and χ^i satisfy the modified trace conditions

$$\mathbb{T}^{(ij}\mathbb{T}^{kl)}\varphi = 0\,, \qquad \mathbb{T}^{(ij}\chi^{k)} = 0\,,$$

where we introduced the notation $\mathbb{T}^{ij} \equiv T^{ij} + \nu\,\delta^{i1}\delta^{j1}$. Young symmetry and spin weight conditions take then the form

$$N_\alpha{}^\beta \varphi = 0 \quad \text{at} \quad \alpha < \beta \qquad \text{and} \qquad N_\alpha \varphi = s_\alpha \varphi\,,$$

and

$$N_\alpha{}^\beta \chi^\gamma + \delta^\gamma_\alpha \chi^\beta = 0 \quad \text{at} \quad \alpha < \beta \qquad \text{and} \qquad N_\alpha \chi = s_\alpha \chi\,, \quad N_\alpha \chi^\alpha = (s_\alpha - 1)\chi^\alpha$$

Introducing operator Z via $\Omega_{-1} \equiv -\frac{\partial}{\partial c_0} Z$ the original triplet equations $\Omega \Psi^{(0)} = 0$ can be cast into the form

$$\Box \Phi - \Omega_0 C = 0 \, , \qquad \Omega_0 \Phi - Z C = 0$$

It follows that C is an auxiliary field and, therefore, using the second equation it can be expressed in terms of $\Omega_0 \Phi$. In other words, C is given by derivatives of Φ, while Φ itself is reduced to the lowest component φ. We arrive at

$$\Box \varphi - (D_i^\dagger + \mu_i) C^i = 0 \, , \qquad D^i \varphi - (D_j^\dagger + \mu_j) \Phi^{i|j} - C^i = 0 \, ,$$

where the component $\Phi^{i|j}$ can be expressed via φ by virtue of the deformed double trace conditions as $\Phi^{i|j} = \frac{1}{2} \mathbb{T}^{ij} \varphi$. Eliminating the auxiliary field C^i we finally arrive at the reduced equations of motion

$$\left[\Box - (D_i^\dagger + \mu_i) D^i + \frac{1}{2} (D_i^\dagger + \mu_i)(D_j^\dagger + \mu_j)(T^{ij} + \nu^{ij}) \right] \varphi = 0 \, ,$$

which are invariant with respect to the gauge transformations

$$\delta \varphi = (D_i^\dagger + \mu_i) \chi^i \, .$$

Here, fields and gauge parameters are subject to the algebraic conditions. Note that setting $\mu, \nu = 0$ we reproduce the Labastida formulation.

Scalar continuous spin case

Let us choose $n = 1$. In this case, all spin weights vanish $s_i = 0$, $i = 2, \ldots$ The reduced equations of motion take the form (Bekaert, Mourad 2005)

$$\Box\varphi - (D^\dagger + \mu)D\varphi + \frac{1}{2}(D^\dagger + \mu)^2(T + \nu)\varphi = 0 \,, \qquad \delta\varphi = D^\dagger\epsilon + \mu\epsilon$$

supplemented with the deformed trace conditions

$$(T + \nu)^2\varphi = 0 \,, \qquad (T + \nu)\epsilon = 0$$

- Note that there are no spin weight conditions in this case. However, the dynamics cannot be restricted to the spin-s subspace since the deformed trace constraints are incompatible with the spin-s weight condition $N\phi = s\phi$.
- Sending both ν and μ to zero we reproduce a sum of the Fronsdal equations for all integer spins.

The deformed trace conditions can be explicitly solved in terms of tensors subjected to the standard trace conditions

$$\varphi = \sum_{n,m=0}^{\infty} \beta_{m,n}(T^\dagger)^m\varphi_{(n)} \,, \qquad \epsilon = \sum_{n,m=0}^{\infty} \beta_{m,n+1}(T^\dagger)^m\epsilon_{(n)} \,,$$

where the rank-n tensors on the right-hand sides satisfy the Fronsdal conditions

$$T^2\varphi_{(n)} = 0 \,, \qquad T\epsilon_{(n)} = 0 \,,$$

Fronsdal basis. The original φ and ϵ are replaced now by infinite collections of Fronsdal (single and double traceless) tensors of ranks running from zero to infinity.

Schuster-Toro representation

It can be explicitly shown that in the Fronsdal basis the metric-like equations take the Schuster-Toro form ($d = 4$: Schuster, Toro 2014, $\forall d$: Metsaev 2016)

$$-\Box\varphi_{(n)} + D^\dagger G_{(n-1)} + \mu\left[G_{(n)} + d_n\, T^\dagger G_{(n-2)}\right] = 0\,, \qquad n = 0, 1, 2, ..., \infty$$

Here,

$$G_{(n)} = A_{(n)} + \mu\, c_n B_{(n)}\,,$$

with the derivative and algebraic terms combined into

$$A_{(n)} = D\varphi_{(n+1)} - \frac{1}{2}D^\dagger\, T\varphi_{(n+1)}\,, \qquad B_{(n)} = \varphi_{(n)} + a_n\, T^\dagger T\varphi_{(n)} + b_n\, T\varphi_{(n+2)}\,,$$

where the coefficients are given by

$$a_n = -\frac{1}{2d + 2n - 8}\,, \qquad b_n = \frac{d + 2n - 2}{2\nu}\,,$$

$$c_n = -\frac{1}{2b_n}\,, \qquad d_n = -\frac{\nu}{(d + 2n - 4)(d + 2n - 6)}\,.$$

We note that $A_{(n)}$ and $B_{(n)}$ as well as $G_{(n)}$ are traceless. These combinations of fields and their derivatives are convenient to build the double-traceless operator $G_{(n)}$.
The gauge transformation reads

$$\delta\varphi_{(n)} = D^\dagger\epsilon_{(n-1)} + \mu\left[\epsilon_{(n)} + d_n\, T^\dagger\epsilon_{(n-2)}\right]\,.$$

This is the Stueckelberg-like transformation law with three different rank traceless gauge parameters, which is typical for massive higher spin theories (Zinoviev 2001).

Light-cone formulation

The quartet grading is defined by $(a = \pm, m)$

$$\deg a_i^{\pm} = \pm 2 \,, \qquad \deg a_i^m = 0 \,, \qquad \deg c_0 = 0 \,, \qquad \deg c_i = 1 \,, \qquad \deg b^i = -1 \,.$$

The triplet BRST operator decomposes as $\Omega = \Omega_{-1} + \Omega_0 + \Omega_1 + \Omega_2 + \Omega_3$, where

$$\Omega_{-1} = p^+ \left(c_i \frac{\partial}{\partial a_i^+} + a_i^- \frac{\partial}{\partial b_i} \right) \,, \qquad \Omega_0 = c_0 (2p^+ p^- + p_m p^m) \,,$$

$$\Omega_1 = c_i p^m \frac{\partial}{\partial a_i^m} + p^+ a_i^- \frac{\partial}{\partial b_i} + \mu \frac{\partial}{\partial b} \,, \quad \Omega_2 = -c_i \frac{\partial}{\partial b_i} \frac{\partial}{\partial c_0} \,, \quad \Omega_3 = p^- (c_i \frac{\partial}{\partial a_i^-} + a_i^+ \frac{\partial}{\partial b_i}) \,.$$

We find $H^0(\Omega_{-1}) = \{\phi(x|a_i^m)\}$, i.e. these are $o(d-2)$ tensors. The reduced BRST charge reads

$$\tilde{\Omega} = c_0 (2p^+ p_- + p^m p_m) \equiv c_0 \Box$$

The light-cone off-shell constraints are given by

$$(\tilde{T} + \nu)\phi = 0 \,, \qquad \tilde{T}^{\alpha} \phi = 0 \,, \qquad \tilde{T}^{\alpha\beta} \phi = 0 \,,$$

$$\tilde{N}_{\alpha}{}^{\beta} \phi = 0 \quad \alpha < \beta \,, \qquad \tilde{N}_{\alpha} \phi = s_{\alpha} \phi \,, \qquad \alpha, \beta = 2, \ldots, n \,,$$

where

$$\tilde{T}^{ij} = \frac{\partial^2}{\partial a_i^m \partial a_{j\,m}} \,, \qquad \tilde{N}_{\alpha}{}^{\beta} = a_{\alpha}^m \frac{\partial}{\partial a^{\beta\, m}} \,, \qquad \tilde{N}_{\alpha} = a_{\alpha}^m \frac{\partial}{\partial a_{\alpha}^m}$$

Light-cone symmetry

Poincare algebra. The Poincare generators in the light-cone basis split into two groups: kinematical $G_{kin} = (P^+, P^m, M^{+m}, M^{+-}, M^{mk})$ and dynamical $G_{dyn} = (P^-, M^{-k})$. After quartet reduction both types of generators act in the subspace, $\tilde{G}_{kin}$ and $\tilde{G}_{dyn}$. We find out that the reduced kinematical generators $\tilde{G}_{kin}$ take the standard form, while the reduced dynamical generators $\tilde{G}_{dyn}$ are given by

$$\tilde{P}^- = -\frac{p^k p_k}{2p^+} \,, \qquad \tilde{M}^{-m} = -\frac{\partial}{\partial p^+} p^m - \frac{\partial}{\partial p_m} \frac{p^k p_k}{2p^+} + \frac{1}{p^+}(S^{mk} p_k + H^m) \,,$$

where S^{mn} and H^m read

$$S^{mn} = a^m_\alpha \frac{\partial}{\partial a^\alpha_n} + a^m \frac{\partial}{\partial a_n} - (m \leftrightarrow n) \,, \qquad H_n = \mu \frac{\partial}{\partial a^n} \,.$$

The elements S^{kl} and H^n satisfy the $iso(d-2)$ commutation relations

$$[S^{kl}, S^{ps}] = \delta^{kp} S^{ls} + 3 \text{ terms} \,, \qquad [S^{kl}, H^n] = \delta^{kn} H^l - \delta^{ln} H^k \,, \qquad [H^k, H^l] = 0 \,.$$

Casimir operators. We immediately see that the $iso(d-2)$ Casimir operators are given by

$$c_2 \equiv H^2 \approx \mu^2 \nu \,,$$

$$c_4 \equiv H^2 S^2 - 2(HS)^2 \approx \mu^2 \nu \sum_{\alpha=2}^{n} s_\alpha (s_\alpha + d - 2\alpha - 3) \,,$$

where $H^2 = H^m H_m$, $S^2 = S_{mn} S^{mn}$, $(HS)^m = H_n S^{nm}$.

Continuous spin-s case

Let us analyze the continuous spin representation labeled by $(s, 0, ..., 0)$ in more detail. In this case there are two oscillators (a, a_1^m) and the trace constraints read

$$(\tilde{T} + \nu)\phi = 0 \,, \qquad \tilde{T}^1\phi = 0 \,, \qquad \tilde{T}^{11}\phi = 0 \,,$$

where

$$\phi = \sum_{p=0}^{\infty} \phi_{m_1 \ldots m_p | n_1 \ldots n_s} \; a^{m_1} \cdots a^{m_p} a_1^{n_1} \cdots a_1^{n_s} \,,$$

and the spin weight condition $\tilde{N}_1 \phi = s\phi$ has been taken into account.
Let $Y(k, l)$ denote a traceless $o(d-2)$ tensor associated to the Young diagram with k indices in the first row and l indices in the second row. Then, the solution is given by

$$\phi \;: \qquad \bigoplus_{l=0}^{s} \bigoplus_{k=s}^{\infty} Y(k, l)$$

- When $s = 0$ the above space is an infinite chain of totally symmetric $o(d-2)$ traceless tensors (Schuster, Toro 2014, Metsaev 2016, 2017).
- For $s \neq 0$ the space is a light-cone version of the covariant formulation discussed in (Zinoviev 2017).
- Let $d = 5$: using the the Hodge duality $Y(k, 1) \sim Y(k, 0)$ and $Y(k, m) = 0$ at $m > 1$ we find out the representation space described in (Brink et al 2002, Metsaev 2017), i.e. two infinite chains of traceless $o(3)$ tensors $Y(k, 0)$ with $k = s, s+1, ..., \infty$.

Final comments

Conclusions

- Implementing differential constraints via the BRST operator and imposing algebraic constraints directly we arrive at the triplet formulation for continuous spin. The resulting equations of motion have a simple form even in the general mixed-symmetry case.
- Using the homological reductions of the triplet BRST operator we found the metric-like formulation that generalizes the Schuster-Toro description of the scalar continuous spin fields. On the other hand, the resulting metric-like formulation is the μ-deformation of the Labastida equations.
- Applying the so-called quartet mechanism we can get rid of the unphysical components of the oscillators to obtain the light-cone form of the continuous spin dynamics. In particular, we explicitly built the $iso(d-2)$ Wigner little algebra and computed its second and fourth Casimir operators.
- There is a functional class so that the gauge symmetry does not kill all PDoF. We demonstrate by performing the light-cone analysis that the system indeed propagates correct degrees of freedom.

Outlooks

- Fermions, SUSY (forthcoming paper with M. Grigoriev and A. Chekmenev)
- Understand group-theoretical meaning of continuous spin fields in AdS
- AdS/CFT correspondence for continuous spin fields...