Deformations of $AdS \times S$ sigma models and 2d dualities

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Integrable deformations of $AdS_n \times S^n$ supercoset string models:

• new aspects of string solutions or conformal sigma models

• new integrable σ-models: apply special deformations, apply 2d dualities (T-duality and its generalizations)

• gauge theory duals? deformations of $N = 4$ SYM?

recently studied examples:

η-model : JJ-type ("Yang-Baxter") deformation of supercoset model [Klimcik; Delduc, Magro, Vicedo; Yoshida et al]

$$L = M_{ab}(g;\eta)\, J_i^a J_i^b, \qquad J = g^{-1}dg$$

λ-model : WZW-type deformation of non-abelian dual of supercoset model [Sfetsos; Hollowood, Miramontes, Schmidtt]

$$L = L_{\rm gWZW}(g', A) + \lambda\, AA \;\rightarrow\; M'_{ab}(g';\lambda)\, J_i'^a J_i'^b$$

• both classically integrable and κ-symmetric GS actions

• UV-finite (scale invariant) sigma models

• which are Weyl-invariant and define critical string models

i.e. corresponding 10d backgrounds solve IIB sugra eqs?

• models related by kind of 2d duality?

related to $AdS_5 \times S^5$ coset by generalized (non-abelian) T-duality?

η-model backgrounds:

in general solve "generalized" supergravity equations

$\star$ equivalent to scale invariance σ-model conditions

[Arutyunov, Frolov, Hoare, Roiban, AT 15]

$\star$ equivalent to κ-symmetry conditions of GS σ-model

[AT, Wulff 16]

Summary:

$\star$ η_0-model based on classical YB R-matrix corresponding to to quasi-Frobenius subgroup H of original symmetry group G

• related to original undeformed model (e.g. $AdS_5 \times S^5$) by non-abelian 2d duality wrt (centrally extended) H (TsT duality in the case of abelian R-matrix) [Hoare, AT 16]

• is Weyl-invariant, i.e. solves standard sugra eqs if H is unimodular ($f^a{}_{ab} = 0$) [Borsato, Wulff 16]

$\star$ η_1-model based on modified classical YB R-matrix

• related to Weyl-invariant λ-model by Posson-Lie duality (generalization of non-abelian duality) [Vicedo 15; Hoare, AT 15; Sfetsos et al 15]

• is only scale-invariant, i.e. solves generalized sugra eqs [Arutyunov, Frolov, Hoare, Roiban, AT 15]

reason: similar $f^a{}_{ab} \neq 0$ obstruction as in non-abelian duality

Review: generalized supergravity equations

string σ-model : $\int d^2z\sqrt{g}(g^{ab}G_{mn}+\epsilon^{ab}B_{mn})\partial_a x^m\partial_b x^n$

• scale invariance or on-shell UV finiteness: $\int d^2z\sqrt{g}\, T^a_a = 0$

$\beta^G_{mn} \equiv R_{mn} - \frac{1}{4}H_{mkl}H_n{}^{kl} = -\nabla_m X_n - \nabla_n X_m$

$\beta^B_{mn} \equiv \frac{1}{2}\nabla^k H_{kmn} = X^k H_{kmn} + \partial_m Y_n - \partial_m Y_n$

X_m and Y_m: freedom of reparams and B-field gauge transf

drop out on eqs of motion or absorbed in $x^m \to x^m + X^m \log\epsilon$

• consistency of string theory: decoupling of conf factor of g_{ab}

Weyl invariance: $T^a_a = 0$ (for ghost-free spectrum, etc.)

• scale invariance $\neq$ 0 conformal invariance in 2d string context

(non-compact σ-model target; non-unitary time-like direction)

• Weyl inv: requires adding $\int d^2z\sqrt{g}R^{(2)}\phi(x)$

to cancel Weyl anomaly if $X_m = \partial_m\phi$, $\quad Y_m = \partial_m\psi$

Weyl invariance conditions:

$$\bar\beta^G_{mn} \equiv R_{mn} - \tfrac{1}{4} H_{mkl} H_n{}^{kl} + 2\nabla_m \nabla_n \phi = 0$$
$$\bar\beta^B_{mn} \equiv \tfrac{1}{2} \nabla^k H_{kmn} - \partial^k \phi H_{kmn} = 0$$

imply "central charge" identity $\partial_m \bar\beta^\phi = 0$ or dilaton equation
$$\bar\beta^\phi \equiv R - \tfrac{1}{12} H^2_{mnk} + 4\nabla^2 \phi - 4(\partial_m \phi)^2 = \text{const}$$

equivalent to string eqs from $S \sim \int d^d x \sqrt{G}\, e^{-2\phi} \bar\beta^\phi$
– same as NS-NS sector of type II sugra equations

Generalized IIB eqs: [Arutyunov,Frolov,Hoare,AT,Roiban]

$$R_{mn} - \frac{1}{4} H_{mkl} H_n{}^{kl} - \mathrm{T}_{mn} + \nabla_m X_n + \nabla_n X_m = 0$$

$$\frac{1}{2} \nabla^k H_{kmn} + \mathcal{I}_{mn} - X^k H_{kmn} - \partial_m X_n + \partial_n X_m = 0$$

$$R - \frac{1}{12} H^2_{mnk} + 4 \nabla^m X_m - 4 X^m X_m = 0$$

$$\mathrm{T}_{mn} \equiv \frac{1}{2} \mathcal{F}_m \mathcal{F}_n + \frac{1}{4} \mathcal{F}_{mpq} \mathcal{F}_n{}^{pq} + \frac{1}{4 \times 4!} \mathcal{F}_{mpqrs} \mathcal{F}_n{}^{pqrs} - \frac{1}{2} G_{mn} \big(\frac{1}{2} \mathcal{F}_k \mathcal{F}^k + \frac{1}{12} \mathcal{F}_{kpq} \mathcal{F}^{kpq} \big)$$

$$\mathcal{I}_{mn} \equiv \frac{1}{2} \mathcal{F}^k \mathcal{F}_{kmn} + \frac{1}{12} \mathcal{F}_{mnklp} \mathcal{F}^{klp}$$

$$\nabla^m \mathcal{F}_m - \mathrm{X}^m \mathcal{F}_m - \frac{1}{6} H^{mnp} \mathcal{F}_{mnp} = 0 \,, \qquad I^m \mathcal{F}_m = 0$$

$$(d\mathcal{F}_1 - \mathrm{X} \wedge \mathcal{F}_1)_{mn} - I^p \mathcal{F}_{mnp} = 0$$

$$\nabla^p \mathcal{F}_{pmn} - \mathrm{X}^p \mathcal{F}_{pmn} - \frac{1}{6} H^{pqr} \mathcal{F}_{mnpqr} - (I \wedge \mathcal{F}_1)_{mn} = 0$$

$$(d\mathcal{F}_3 - \mathrm{X} \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{mnpq} - I^r \mathcal{F}_{mnpqr} = 0$$

$$\nabla^r \mathcal{F}_{rmnpq} - \mathrm{X}^r \mathcal{F}_{rmnpq} + \frac{1}{36} \varepsilon_{m \dots w} H^{rst} \mathcal{F}^{uvw} - (I \wedge \mathcal{F}_3)_{mnpq} = 0$$

$$(d\mathcal{F}_5 - \mathrm{X} \wedge \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{mnpqrs} + \frac{1}{6} \varepsilon_{m \dots w} I^t \mathcal{F}^{uvw} = 0$$

two vectors X_m and I_m satisfy

$$X_m = \mathrm{X}_m + I_m \ , \qquad \nabla_m I_n + \nabla_n I_m = 0 \ , \qquad \mathrm{X}^m I_m = 0$$

$$\partial_m \mathrm{X}_n - \partial_n \mathrm{X}_m + H_{mnk} I^k = 0 \quad \to \quad \mathrm{X}_m = \partial_m \phi + B_{mn} I^n$$

• I_m= Killing vector , X_m= generalized "dilaton gradient"
(solving generalized "dilaton Bianchi identity")

• if background has no Killings, i.e. $I_m = 0$ then $\mathrm{X}_m = \partial_m \phi$
and generalized eqs become standard type IIB supergravity eqs

• if G admits Killing: choose I_m = Killing and get new soln.
corresponding to κ-symmetric, scale-invariant
but not Weyl-invariant GS sigma model

• thus differ by "measure zero" set of special symmetric solns
corresponding to scale but not Weyl inv σ-models

• these special solutions of generalized eqs are classically
T-dual to type II sugra solutions with non-isometric dilaton

Review of 2d dualities

Abelian T-duality:

$$L = G(y)\partial x \partial x \to G(y)(\partial x + A)^2 + \widetilde{x} F(A)$$

(i) gauge $x = 0$, integrate out A_a, get dual action $\widetilde{L} = G^{-1}\partial\widetilde{x}\partial\widetilde{x}$

(ii) gauge $A_1 = 0$, integrate A_0, get "doubled" action: [AT 93]

$$L(x, \widetilde{x}) = 2\dot{x}\widetilde{x}' - Gx'x' - G^{-1}\widetilde{x}'\widetilde{x}'$$

cf. $L = p\dot{x} - H(p, x), \quad p = \widetilde{x}'$

general case with $G_{ij} \to G_{ij} + B_{ij} \quad (i = 1, ..., d)$

$$L = \eta_{mn}\dot{X}^m X'^n - M_{mn} X'^m X'^n \ , \qquad X = (x^i; \widetilde{x}^i)$$

$$\eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ , \quad M = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

$O(d, d)$ sym: $X \to \Lambda X, \ M \to \Lambda^T M \Lambda, \quad M \in \frac{O(d,d)}{O(d)\times O(d)}$

• special case of $O(2,2)$ is TsT duality:

$$x_1 \to \widetilde{x}_1; \quad x_2 \to x_2 + \gamma \widetilde{x}_1; \quad \widetilde{x}_1 \to x_1$$

$$ds^2 = dy^2 + f_1(y) dx_1^2 + f_2(y) dx_2^2$$

$$d\widetilde{s}^2 = dy^2 + U(y)\big[f_1(y) dx_1^2 + f_2(y) dx_2^2\big]$$

$$e^{2\phi} = U(y) \equiv \frac{1}{1 + \gamma^2 f_1(y) f_2(y)}\,, \qquad B_{y_1 y_2} = \gamma f_1 f_2 U$$

• interpolates original ($\gamma = 0$) and double-T-dual ($\gamma = \infty$)

• used to construct useful examples of supergravity backgrounds: Melvin, "non-commutative", "β-deformation" [Lunin,Maldacena 05]

• can be interpreted as non-abelian dual for centrally extended 2d Heisenberg group [Hoare, AT]

$\star$ T-duality preserves Weyl invariance of sigma-model (with shifted dilaton: $e^{-2\phi}\sqrt{G} = e^{-2\widetilde{\phi}}\sqrt{\widetilde{G}}, \quad \widetilde{G} = G^{-1}$)

e.g. maps one isometric supergravity solution into another

Non-abelian duality

can apply when sigma-model has global symmetry G

(abelian T-duality is special case when $G = [U(1)]^d$)

• simplest example: PCM $L = J^2, \quad J = g^{-1}\partial g$

$$L(g, v, A) = (g^{-1}\partial g + g^{-1}Ag)^2 + vF(A)$$

$F = dA + A \wedge A$, $\quad v \in \mathrm{alg}(G)$ is analog of $\widetilde{x}$ if $g = e^{x^a T_a}$

(i) gauge $g = 1$ and integrate $A_a \to$ dual action $L(v)$

(generically with no manifest isometries)

(ii) gauge $A_1 = 0$ and integrate $A_0 \to L(g, v)$: analog of $L(x, \widetilde{x})$:

• NAD is path integral transform: [Fridling, Jevicki; Fradkin, AT 84]

preservation of Weyl invariance / supergravity equations?

yes, if $f^b{}_{ba} = 0$ (G is "unimodular") [Alvarez,Alvarez-Gaume,Lozano; Elitzur,Giveon,Rabinovici,Schwimmer,Veneziano 94]

[same as condition of consistency of Scherk-Schwarz compactification: group invariance of the volume]

• Example: non-semisimple dilatation+translations algebra

$[D, P_a] = P_a, \ [P_a, P_b] = 0$ has $n_a \equiv f^b_{ba} \neq 0$

for $n_a \neq 0$ NAD may map Weyl-inv model to scale-inv one, i.e. supergravity solution to solution of "generalized" equations

• on curved 2d background in conformal gauge $g_{ij} = e^{2\sigma}\eta_{ij}$

$$L = E_{ab}(y)\, J^a_+ J^b_- + \ldots$$

$$J^a_i = \mathrm{tr}(g^{-1}\partial_i g\, \widetilde{T}^a) \ , \qquad [T_a, T_b] = f^c{}_{ab} T_c \ , \qquad \mathrm{tr}(T_a \widetilde{T}^b) = \delta^b_a$$

$\widetilde{T}^a$ is a dual algebra basis

dilaton term $\frac{1}{4\pi}\int d^2 z \sqrt{g} R^{(2)} \phi(y) = -\frac{1}{2\pi}\int d^2 z\, \phi(y)\, \partial^2 \sigma$

original action from path integral over $A^a_\pm$ and v_a with

• "interpolating" action ($g = 1$ gauge)

$$\widehat{L} = E_{ab}(y)\, A_+^a A_-^b + \ v_a \epsilon^{ij} F_{ij}^a + 2\alpha' \sigma\, n_a \partial^i A_i^a - 2\alpha' \phi(y)\, \partial_+ \partial_- \sigma$$

$$F_{ij}^a \equiv \partial_i A_j^a - \partial_j A_i^a + f^a_{\ bc} A_i^b A_j^c\ , \qquad\qquad n_a = \mathrm{tr}\, T_a = f^b_{\ ba}$$

• n_a-term: from Jacobian of transformation from $A_+ = h^{-1}\partial_+ h,\ A_- = h'^{-1}\partial_- h'$ to h, h' in curved space related to the "mixed" gauge anomaly $\partial_i \frac{\delta\Gamma}{\delta A_i} \sim R^{(2)}$ corresponding to $\nabla^i A_i \nabla^{-2} R$ term in eff. action

• dual model found by integrating out A_i^a

$$\widetilde{L} = N^{ab}(v, y)\big(\partial_+ v_a + \alpha' n_a \partial_+ \sigma\big)\big(\partial_- v_b - \alpha' n_b \partial_- \sigma\big)$$
$$- 2\alpha' \big(\phi(y) + \tfrac{1}{2} \ln \det N\big)\, \partial_+ \partial_- \sigma$$

$$N^{ab} = \big[E_{ab}(y) - v_c f^c_{\ ab}\big]^{-1}$$

• If $n_a \neq 0$ is not a local σ-model on a curved 2d background

corresponding dual target space background ($\widetilde{G}$, $\widetilde{B}$, $\widetilde{\phi}$)
will not solve critical string Weyl-invariance conditions

• on flat 2d backgr NAD dual σ-model will still be scale inv:
$\widetilde{G}, \widetilde{B}$ in superstring context will solve generalized sugra eqs

• scale-invariant dual background $(\widetilde{G}, \widetilde{B})$ can be naturally associated to solution of Weyl-invariance conditions by applying formal (2d flat space) T-duality in isometric part of v_a parallel to n_a

[set $v_a = u_a + n_a z$, then z is isometry of dual background

$v_a F^a_{ij} = u_a F^a_{ij} + z(\partial_i \mathrm{A}_j - \partial_j \mathrm{A}_i)$, $\mathrm{A}_i = n_a A^a_i = \mathrm{tr} A_i$

$(n_a f^a_{\ bc} = f^d_{\ da} f^a_{\ bc} = 0$ from Jacobi identity)

$v_a \epsilon^{ij} F^a_{ij} + 2\alpha' \sigma\, n_a \partial^i A^a_i \to -2\epsilon^{ij} \partial_i z \mathrm{A}_j + 2\alpha' \sigma\, \partial^i \mathrm{A}_i$

T-duality in the z direction gives $\mathrm{A}_i = \partial_i \widetilde{z}$ and

$2\alpha' \sigma\, \partial^i \mathrm{A}_i \to -2\alpha' \widetilde{z}\, \partial_+ \partial_- \sigma \ \sim \widetilde{z} \sqrt{g} R^{(2)}$]

get dilaton linear in dual coordinate: $\phi \sim \widetilde{z}$

• integrating rest of A_i^a out gives local Weyl-invariant σ-model

with a linear non-isometric dilaton:

solution of supergravity which cannot be T-dualized in $\widetilde{z}$

to find another solution;

• thus by formal (“flat-space”) T-duality

get solution of generalized eqs

TsT as special case of non-abelian duality

start with $\mathcal{L} = f_1(\partial_i x_1)^2 + f_2(\partial_i x_2)^2$

"interpolating" action for first 2 steps of TsT

$$\widetilde{\mathcal{L}} = f_1(A_i)^2 + f_2(\partial_i x_2 + \gamma \partial_i \widetilde{x}_1)^2 + 2\epsilon^{ij}\partial_i \widetilde{x}_1 A_j$$

T-dualizing again $\widetilde{x}_1 \to x_1$ by introducing A'_i

$$\mathcal{L}' = f_1(A_i)^2 + f_2(\partial_i x_2 + \gamma A'_i)^2 + 2\epsilon^{ij} A'_i A_j + 2\epsilon^{ij}\partial_i x_1 A'_j$$

after field redefenitions ($x_i \to v_i$) and rescalings

$$\mathcal{L}' = f_1(A^1_i)^2 + f_2(A^2_i)^2 + 2\epsilon^{ij}\left(v_1 \partial_i A^1_j + v_2 \partial_i A^2_j\right) - 2\gamma^{-1}\epsilon^{ij} A^1_i A^2_j$$

if first integrate over v_r get $A^r_i = \partial_i x^r_i$ i.e. original model

integrating A, A' get σ-model corresponding to TsT background

$\star$ can get same model as NAD for 2d Heisenberg algebra centrally-extended 2d translations = Heisenberg algebra

$$T_a = (P_1, P_2, Z)\ , \qquad [P_r, P_s] = \epsilon_{rs} Z\ , \qquad [P_r, Z] = 0$$

$$v_a \epsilon^{ij} F^a_{ij} = 2\epsilon^{ij}\left[v_1 \partial_i A^1_j + v_2 \partial_i A^2_j + v_3(\partial_i A^3_j + A^1_i A^2_j)\right]$$

v^3 and A^3_i correspond to central generator Z

integrating A^3_i gives $v_3 = \mathrm{const} \sim \gamma^{-1} \to$ TsT model

• thus considering central extension allows
to introduce an extra free parameter γ absent in NAD

• γ as background value of dual coordinate
corresponding to the central generator Z

• origin of B-field of the resulting background traced
to the non-abelian nature of the Heisenberg algebra

• related to the non-commutativity of dual gauge theory
(cf. [Hashimoto, Itzhaki; Maldacena, Russo; Lunin, Maldacena])

Poisson-Lie duality [Klimcik, Severa 95]

generalization of non-abelian duality: replace isometry by weaker condition of Poisson-Lie symmetry

• special choice of non-constant kinetic matrix $E_{ab}(g)$ in

$$L = E_{ab}(g) J^a_+ J^b_- = K_{ij}(x)\partial_+ x^i \partial_- x^j , \quad K = G + B$$

$$\delta g = \epsilon g , \quad g = g(x), \quad \epsilon = \epsilon^a T_a \, , \quad \delta x^i = \mathrm{R}^i_a(x)\epsilon^a$$

$$[\mathcal{L}_{\mathrm{R}_a}, \mathcal{L}_{\mathrm{R}_b}] = f_{ab}{}^c \mathcal{L}_{\mathrm{R}_c} , \qquad [T_a, T_b] = f^c_{ab} T_c$$

instead of G-invariance $\mathcal{L}_{\mathrm{R}_a} K_{ij} = 0$ demand

$$\mathcal{L}_{\mathrm{R}_a} K_{ij} = K_{ik} \mathrm{R}^k_b \widetilde{f}^{bc}{}_a \mathrm{R}^m_c K_{mj}$$

$\widetilde{f}^{bc}{}_a$ – structure consts of dual group $\widetilde{G}$ of same dimension

• current $J_a = K_{ij} dx^i \mathrm{R}^j_a$: “non-commutative conservation”

$$d * J_a + \tfrac{1}{2}\widetilde{f}^{bc}{}_a * J_b \wedge * J_c = 0$$

• Poisson-Lie duality: $E_{ab}(g) J^a_+ J^b_- \to \widetilde{E}^{ab}(\widetilde{g}) \widetilde{J}_{a+} \widetilde{J}_{b-}$

exchanges roles of groups G and $\widetilde{G}$

Drinfeld double: Lie algebra $\mathcal{D} = \mathrm{alg}(G) \oplus \mathrm{alg}(\widetilde{G})$
sum of two maximally isotropic ($\langle X|Y\rangle = 0$) subalgebras wrt
bilinear form: $\langle T_a|T_b\rangle = \langle \widetilde{T}^a|\widetilde{T}^b\rangle = 0\,, \quad \langle T_a|\widetilde{T}^b\rangle = \delta_a{}^b$

$$[T_a, T_b] = f_{ab}^c T_c, \qquad [\widetilde{T}^a, \widetilde{T}^b] = \widetilde{f}^{ab}{}_c \widetilde{T}^c$$

$$[T_a, \widetilde{T}^b] = \widetilde{f}^{bc}{}_a T_c - f_{ac}{}^b \widetilde{T}^c$$

$[T_A, T_B] = f_{AB}^C T_C, \quad T_A = (T_a, \widetilde{T}^a),$ Jacobi:

$$\widetilde{f}^{a[c}{}_{[b} f_{d]a}{}^{g]} - \widetilde{f}^{gc}{}_a f_{db}{}^a = 0$$

- two PL dual σ-models: data – groups G and $\widetilde{G}$ and E_{0ab}=const

$$L = E_{ab}(g)\, J_+^a J_-^b\,, \qquad J^a = \mathrm{tr}(T^a g^{-1} dg)\,, \qquad E = [E_0^{-1} + \Pi(g)]^{-1}$$

$$\widetilde{L} = \widetilde{E}^{ab}(\widetilde{g})\, \widetilde{J}_{a+} \widetilde{J}_{b-}\,, \quad \widetilde{J}_a = \mathrm{tr}(\widetilde{T}_a \widetilde{g}^{-1} d\widetilde{g})\,, \qquad \widetilde{E} = [E_0 + \widetilde{\Pi}(\widetilde{g})]^{-1}$$

$$g \in G, \;\; \{T_a\}, \qquad \widetilde{g} \in \widetilde{G}, \;\; \{\widetilde{T}^a\}, \qquad a = 1, ..., d$$

E_0=const; Π,$\widetilde{\Pi}$ antisymm: $\Pi^{ab} = V^{ca} U_c{}^b, \ \ \widetilde{\Pi}_{ab} = \widetilde{V}_{ca} \widetilde{U}^c{}_b$

$$g^{-1} T_a g = V_a{}^b(g) T_b \, , \qquad g^{-1} \widetilde{T}^a g = U^{ab}(g) T_b + (V^{-1})_b{}^a(g) \widetilde{T}^b$$

$$\widetilde{g}^{-1} \widetilde{T}^a \widetilde{g} = \widetilde{U}^a{}_b(\widetilde{g}) \widetilde{T}^b \, , \quad \widetilde{g}^{-1} T_a \widetilde{g} = \widetilde{V}_{ab}(\widetilde{g}) \widetilde{T}^b + (\widetilde{U}^{-1})^b{}_a(\widetilde{g}) T_b$$

• PL duals are canonically equivalent : $\widetilde{P}^a = J^a_0, \ \ \widetilde{J}_{a0} = P_a$

$J^a_0 = J^a_1 + \Pi^{ab} P_b \, , \ \ \widetilde{J}_{a0} = \widetilde{J}_{1a} + \widetilde{\Pi}_{ab} \widetilde{P}^b$

Examples:

• $\mathcal{D}$ is abelian: G=$\widetilde{G}$= T^d – abelian T-duality

• $\mathcal{D}$ is semi-abelian: $\widetilde{G}$=abelian – non-abelian duality

• $\mathcal{D} = SL(2, C), \quad G = SU(2), \quad \widetilde{G} = B_2$=Borel

e.g. η-deformation of S^2 is PL dual

to corresponding λ-model [Hoare, AT 15]

• generic E-model (parametrized by const matrix E_0)

is PL dual (with $\mathcal{D} = G^{\rm c}$) to gWZW "Q-model" [Klimcik 16]

• PL duality: classical phase space structures of dual models are closely related (Poisson bracket algebra, hidden charges,....)

• PL duality preserves β-functions – scale invariance

[Alexeev, Klimcik, AT 95; Sfetsos et al 98; Klimcik et al 01]

• Weyl invariance (and thus supergravity equations) preserved provided both $f^a_{\ ab} = 0$ and $\widetilde{f}^{ba}_{\ \ a} = 0$

[Tuyrin, Unge 96; Bossard, Mohammedi 01]

• "doubled" action $I(g, \widetilde{g})$ = special action on the double $\mathcal{D}$

[Alekseev, Klimcik, AT 95; Klimcik, Severa 95]

$$
\begin{aligned}
I[\ell] &= I'_{\rm WZW}[\ell] - \int d^2z \langle J_1|\mathcal{R}|J_1\rangle \\
&= \int d^2z \Big(\langle J_0|J_1\rangle - \langle J_1|\mathcal{R}|J_1\rangle\Big) + \text{WZ-term} \\
&\qquad J_r = \ell^{-1}\partial_r \ell \ \in \mathcal{D}
\end{aligned}
$$

$\mathcal{R}$ operator: ± 1 on "chiral" subspaces defined by E_0-matrix:

$$\mathcal{D}^+ = \{T_a + E_{0ab}\widetilde{T}^b\}, \quad \mathcal{D}^- = \{T_a - E_{0ba}\widetilde{T}^b\}, \quad \langle \mathcal{D}^+|\mathcal{D}^-\rangle = 0$$

Integrable sigma models

given integrable string σ-model – how to find another one?

$\star$ if abelian isometries: apply abelian T-duality

e.g. TsT $\to$ β-deformation of $AdS_5 \times S^5$ [Lunin,Maldacena 05]

$\star$ if non-abelian isometries: apply non-abelian duality

$\star$ if PL symmetry: apply PL duality

Other options? e.g. start with PCM and deform

$L = J_+ J_- + M(g) J_+ J_-, \qquad J = g^{-1} dg$

• which $M(g)$ preserves integrability?

• conformal invariance ?

deform WZW or F/F gWZW

"universal WZW" or "Q-model" [AT 93]

$$I_Q(g,A) = I_{\rm gWZW}(A,g) + \int d^2z\, A_+ Q A_- \ , \qquad Q = {\rm const}$$

$$I_Q = I_{\rm WZW}(g) + \int d^2z\, {\rm Tr}\Big[-A_+\partial_- g g^{-1} + g^{-1}\partial_+ g A_- + g^{-1}A_+ g A_- + A_+(Q-I)A_-\Big]$$

$${\rm Tr}[g^{-1}A_+ g A_- + A_+(Q-I)A_-] = A^a_+\Big[C_{ab}(g) + Q_{ab} - \eta_{ab}\Big]A^b_-$$

$$[T_a,T_b] = i f^c{}_{ab} T_c \ , \qquad g^{-1}\partial g = J^a T_a$$

$${\rm Tr}(T_aT_b) = \eta_{ab} \ , \qquad C_{ab}(g) \equiv {\rm Tr}(T_a g T_b g^{-1})$$

$Q_{ab} = {\rm Tr}(T_a Q T_b)$=const matrix of deformation parameters

$$I_Q(g) = I_{\rm WZW}(g) + \int d^2z\, \mathcal{M}_{ab}(g) J^a_+ J^b_-$$

$$\mathcal{M} = [C(g) + Q - I]^{-1}$$

• one-loop Weyl inv: conditions on Q [AT 93]

$$f_{mkl}f_{nkl} - \frac{1}{2}f_{mkl}f_{nk'l'}\mathcal{K}_{kk'}\mathcal{K}_{ll'} - \frac{1}{2}f_{mkl}f_{nk'l'}\mathcal{K}^{-1}_{kk'}\mathcal{K}^{-1}_{ll'}$$
$$+f_{m'kl}f_{n'k'l'}\mathcal{K}_{mm'}\mathcal{K}_{nn'}\mathcal{K}_{kk'}\mathcal{K}^{-1}_{ll'} - f_{m'kl}f_{n'kl}\mathcal{K}_{mm'}\mathcal{K}_{nn'} = 0$$
$$\mathcal{K} \equiv I - 2Q^{-1}$$

similar to Virasoro master equation [Halpern,Kiritsis et al 90]

⋆ require classical integrability:

"λ-model": $\quad Q \sim I, \quad \mathcal{M} \sim \big[I - \lambda\, C(g)\big]^{-1}$

integrable interpolation between

gWZW and non-abelian dual of PCM [Sfetsos 13]

λ-model : coset case $\quad Q \sim P = P_{F/G}$=projector

deformed F/F gauged WZW model; parameters (k, λ):

$\gamma \equiv \lambda^{-2} - 1; \quad g \in F, \quad A \in \text{alg}(F)$

$$I_{k,\lambda}(g, A) = k\Big[I_{\text{gWZW}}(g, A) - \gamma \int d^2z \, \text{Tr}\big(A_+ P A_-\big)\Big]$$

(i) $\gamma \to \infty$: $\quad F/G$ gWZW

(ii) $\gamma \to 0$, $k \to \infty$: $\quad \lambda = 1 - \frac{\pi}{k} h + \ldots, \ \gamma = \frac{2\pi}{k} h + \ldots$

$g = e^{-v/k} = 1 - \frac{1}{k} v + \ldots, \qquad v \in \text{alg}(F)$ [Sfetsos 13]

$I_{k\to\infty,\lambda\to 1} = \int d^2x \, \text{Tr}\big[\, v \, F_{+-}(A) - h \, A_+ P A_- \big]$

• interpolating between F/G gWZW model and non-abelian dual of F/G coset model

• to get conf. invariance: promote to superstring model

η-model : integrable deformation of PCM

"Yang-Baxter" σ-model [Klimcik 02,08]

$$L_0 = h\,\mathrm{Tr}\,(J_+ J_-), \qquad J = g^{-1} dg, \quad g \in G$$

$$L_\eta = h\,\mathrm{Tr}(J_+ M J_-)\,, \qquad M(g) = \frac{1}{1 - \varkappa R_g}\,, \qquad \varkappa = \frac{2\eta}{1-\eta^2}$$

$$R_g(X) = g^{-1} R(g X g^{-1}) g\,, \quad X \in \mathrm{alg}(G), \quad R^T = -R$$

R=const – analog of Q – solution of modified YBE

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = c\,[X, Y]$$

- classical integrability for any $\varkappa$: Lax pair
- manifest $G \times G$ symmetry broken to $[U(1)]^r \times G$
- hidden non-local charges: $U_q[\mathrm{alg}(G)]$, $\; q = e^{-\varkappa/h}$
- simplest example: "squashed" S^3 [Cherednik 81]

η-deformed G/H model [Delduc, Magro, Vicedo 13]

$$L_\eta = h\,\mathrm{Tr}\left(J_+ M J_-\right)\,, \qquad\qquad M = \frac{P}{1-\varkappa R_g\cdot P}$$

$U_q[\mathrm{alg}(G)]$ symmetry: $\quad q = e^{-\varkappa/h}$

- local charges: G broken to $[U(1)]^r$ – Cartans of $\mathrm{alg}(G)$
- non-local charges: for simple roots

$$\{Q_{+\alpha_n}, Q_{-\alpha_m}\} = i\delta_{nm}[C_n]_q\ , \qquad\qquad [C]_q = \frac{q^C-q^{-C}}{q-q^{-1}}$$

- deformed S^2 = “sausage” model [Fateev, Onofri, Zamolodchikov 93]

λ-model and η-model :
actions look very different;
different manifest symmetries but closely related:

• η-model from λ-model as a limit (+ analytic cont) [Hoare, AT 15]

• form pair of Poisson-Lie dual models
share q-deformed phase space structure
[Vicedo 15; Hoare, AT 15; Sfetsos, Siampos,Thompson 15; Klimcik 16]
λ-model – analog of diagonal subgroup member
η-model – analog of solvable subgroup member
– two "faces" of first order "interpolating" theory
defined on the double

$AdS_5 \times S^5$ superstring σ-model: $\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

Z_4 of $\mathfrak{psu}(2,2|4)$: projectors $P_0 + P_2 + P_1 + P_3 = I$

$$L_0 = h\,\mathrm{Str}\big(J_+\, M_0\, J_-\big)$$

$$J_a = g^{-1}\partial_a g\ , \qquad M_0 \equiv P_2 + \tfrac{1}{2}(P_1 - P_3)$$

η_c-deformed model: $c = 1$ or $c = 0$

$$L_\eta = h\,\mathrm{Str}\big(J_+\, M\, J_-\big)\ , \qquad \varkappa = \frac{2\eta}{1-\eta^2}$$

$$M = \frac{P_\varkappa}{1 - \varkappa R_g \cdot P_\varkappa}\ , \qquad P_\varkappa = P_2 + \frac{1}{1+c\sqrt{1+\varkappa^2}}(P_1 - P_3)$$

$$R_g(X) = g^{-1}R(gXg^{-1})g\ , \qquad X \in \mathrm{alg}(\widehat{F})$$

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = c\,[X, Y]$$

classically integrable; fermionic "kappa-symmetry"

$\star$ η_1 model: modified classical YBE

• $R = \pm 1$ on $\pm$ roots, 0 on Cartans [Delduc, Magro, Vicedo 13,14]

• all of $AdS_5 \times S^5$ is deformed; susy broken;

6 Cartan directions of $SO(2,4) \times SO(6)$ remain isometries

• PL dual to λ-model [Vicedo 15]

$\star$ η_0 model: classical YBE [Kawaguchi, Matsumoto, Yoshida 14]

$R(X) = r^{ij}\mathrm{Str}(T_i X) T_j$, $\quad \mathrm{R} = \frac{1}{2} r^{ij} T_i \wedge T_j \in \mathrm{alg}(F) \times \mathrm{alg}(F)$

(i) abelian: $[T_i, T_j] = 0$; (ii) non-abelian (e.g. Jordanian)

• can deform AdS_5 or S^5 separately; some susy and isometries

• abelian models are “trivial”: related by TsT to

undeformed $AdS_5 \times S^5$ [Matsumoto, Yoshida 14; van Tongeren et al 15,16]

• non-abelian models: related to $AdS_5 \times S^5$

by non-abelian duality for qF subgroup corresponding to R

[Hoare, AT; Vicedo; Borasto, Wulff]

η_0-model – YB deformations

• YB deformation based on abelian r-matrix $\leftrightarrow$
TsT transform of undeformed G-model $\leftrightarrow$
NAD wrt centrally-extended abelian algebra

• YB model with non-abelian r-matrix $\leftrightarrow$
NAD wrt centrally-extended quasi-Frobenius subalgebra of G
[Hoare, AT 16; Borsato, Wulff 16]

• YB r-matrices of real semisimple algebra $\leftrightarrow$
quasi-Frobenius subalgebras of G [Stolin; Vicedo]

qF algebra:
(i) equipped with non-degenerate skew bilinear form
$\omega(X,Y) = -\omega(Y,X)$ which is
(ii) 2-cocycle: $\omega([X,Y],Z)+\omega([Y,Z],X)+\omega([Z,X],Y) = 0$
(iii) ω^{-1} satisfies cYBE, i.e. can be identified with r-matrix

Examples:

$\star$ abelian $r = T_1 \wedge T_2 + T_3 \wedge T_4 + ...,\quad [T_a, T_b] = 0$

$\star$ unimodular non-abelian: $[T_a, T_b] = f^c_{ab} T_c\ ,\qquad f^b_{ba} = 0$

$\star$ non-unimodular jordanian r-matrices: $[T_1, T_2] = T_2, ...$

• if qF subalgebra is not unimodular

NAD breaks Weyl invariance

• in the case of $AdS_5 \times S^5$ non-trivial examples

for subgroups of $SO(2,4)$

η_1-model – modified YB deformations

[Klimcik; Delduc, Magro, Vicedo]

• based on solution of modified classical YBE:

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)) = [X, Y]$$

• symmetry of original model is q-deformed

• resulting η-model is scale but not Weyl-inv [Arutyunov et al]

• related by flat-space T-duality to Weyl-inv theory
with linear non-isometric dilaton [Hoare, AT]

• related by PL duality to Weyl-invariant λ-model
[Hollowood, Miramontes, Schmidt; Vicedo, Hoare, AT;

Sfetsos, Siampos, Thompson; Klimcik]

• origin of Weyl anomaly in PL duality should be in $\widetilde{f}^{a}_{ab} \neq 0$

• properly defined theory on a double should be Weyl-invariant?
consistent string theory in "complexified" space?

Conclusions and open questions:

by-products of studies of integrable

$AdS_5 \times S^5$ supercoset σ-model deformations:

• discovery of generalized sugra equations equivalent to conditions of classical $\varkappa$-symmetry of GS σ-model

• role of generalized T-dualities:

non-abelian and Poisson-Lie transformations

implications in AdS/CFT context?

• gauge theory connection unclear in general

apart from "abelian" η_0-model – standard TsT of $AdS_5 \times S^5$

$\rightarrow$ non-commutative versions of SYM / integrable twists

• "non-abelian" cases: non-trivial background fields

vacua breaking 4d Poincare symmetry ...

examples useful from gauge-theory perspective ?