Higher Spin Theory and Holography hSth-5

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Finiteness of the triple ghost-gauge vertices in $\mathcal{N} = 1$ SYM theories

Higher spin theories are invariant under a some symmetry transformations. These transformations can include supersymmetry. For example, superstring theories can be considered as supersymmetric higher spin theories. Absence of ultraviolet divergences in these theories makes them especially interesting.

Even in usual quantum field theory, supersymmetry allows constructing a theory finite in all orders, namely, the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory (SYM). The UV behaviour of other supersymmetric theories is also better than in the non-supersymmetric case due to some non-renormalization theorems.

For $\mathcal{N} = 2$ SYM divergences appear only in the one-loop approximation. $\mathcal{N} = 2$ hypermultiplets are not renormalized.

Absence of higher loop quantum corrections in theories with extended supersymmetry can essentially simplify the theoretical investigation of these theories in some aspects.

Non-renormalization theorems for $\mathcal{N} = 1$ supersymmetry

Although theories with extended supersymmetry have a lot of very interesting features, the low energy physics seems to be $\mathcal{N} = 1$ supersymmetric. Even in this case supersymmetry leads to some non-renormalization theorems.

The most known theorem is that there are no divergent quantum corrections to the superpotential of $\mathcal{N} = 1$ supersymmetric theories.

One more statement is that the β-function of $\mathcal{N} = 1$ SYM is related to the anomalous dimensions of the matter superfields by the so called NSVZ β-function. For pure $\mathcal{N} = 1$ SYM theory it gives the exact expression for the β-function in the form of the geometric series.

Here we argue that in $\mathcal{N} = 1$ SYM theories the three-point ghost-gauge vertices are finite in all orders, where the gauge leg corresponds to the quantum gauge superfield.

Let us consider the $\mathcal{N}=1$ SYM theory described by the action

$$S=\frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\left(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\right)+\text{c.c.}\Big\},$$

where the supersymmetric gauge field strength is defined as

$$W_a=\frac{1}{8}\bar{D}^2\left(e^{-2V}D_ae^{2V}\right).$$

We assume that the theory is invariant under the gauge transformations

$$\phi\to e^A\phi;\qquad e^{2V}\to e^{-A^+}e^{2V}e^{-A},$$

where the parameter $A=ie_0A^BT^B$ is an arbitrary chiral superfield.

Higher covariant derivative regularization

We use the higher covariant derivative regularization

A.A.Slavnov, Nucl.Phys., **B31**, (1971), 301; Theor.Math.Phys. **13** (1972) 1064.

because it is consistent and does not break supersymmetry:

V.K.Krivoshchekov, Theor.Math.Phys. **36** (1978) 745;
P.West, Nucl.Phys. B268, (1986), 113.

It can be also used for theories with $\mathcal{N} = 2$ supersymmetry

V.K.Krivoshchekov, Phys.Lett. B **149** (1984) 128; I.L.Buchbinder, K.S., Nucl.Phys. **B883** (2014) 20; I.L.Buchbinder, N.G.Pletnev, K.S., Phys.Lett. **B751** (2015) 434.

To regularize the theory by higher derivatives, it is necessary to add a term with higher degrees of covariant derivatives to the action. Then divergences remain only in one-loop approximation. These remaining divergences are regularized by inserting the Pauli–Villars determinants into the generating functional.

A.A.Slavnov, Theor.Math.Phys. **33** (1977) 977.

Quantum-background splitting is made by the substitution

$$e^{2V} \to e^{\mathbf{\Omega}^+} e^{2V} e^{\mathbf{\Omega}}.$$

The background superfield $\boldsymbol{V}$ is defined by $e^{2\boldsymbol{V}} = e^{\mathbf{\Omega}^+} e^{\mathbf{\Omega}}$.
We choose the following higher derivative term

$$S_\Lambda = \frac{1}{2e_0^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\, d^2\theta\, e^{\Omega} e^{\mathbf{\Omega}} W^a e^{-\mathbf{\Omega}} e^{-\Omega}\Big[R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)-1\Big]_{Adj}$$
$$\times e^{\Omega} e^{\mathbf{\Omega}} W_a e^{-\mathbf{\Omega}} e^{-\Omega} + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^+ e^{\mathbf{\Omega}^+} e^{\Omega^+}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)-1\Big] e^{\Omega} e^{\mathbf{\Omega}}\phi,$$

and the gauge fixing term

$$S_{\mathrm{gf}} = \frac{1}{e_0^2}\mathrm{tr}\int d^4x\, d^4\theta\,\Big(16\xi_0\, f^+\Big[e^{\mathbf{\Omega}^+} K^{-1}\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)e^{\mathbf{\Omega}}\Big]_{Adj} f$$
$$+ e^{\mathbf{\Omega}} f e^{-\mathbf{\Omega}}\boldsymbol{\nabla}^2 V + e^{-\mathbf{\Omega}^+} f^+ e^{\mathbf{\Omega}^+}\bar{\boldsymbol{\nabla}}^2 V\Big),$$

where the regulators R, F, and K have a rapid growth at infinity.

Actions for the Faddeev–Popov and Nielsen–Kallosh ghosts have the form

$$S_{\mathsf{FP}} = \frac{1}{e_0^2}\mathrm{tr}\int d^4x\, d^4\theta\, \Big(e^{\boldsymbol{\Omega}}\bar{c}e^{-\boldsymbol{\Omega}} + e^{-\boldsymbol{\Omega}^+}\bar{c}^+e^{\boldsymbol{\Omega}^+}\Big)$$
$$\times\Big\{\Big(\frac{V}{1-e^{2V}}\Big)_{Adj}\Big(e^{-\boldsymbol{\Omega}^+}c^+e^{\boldsymbol{\Omega}^+}\Big) + \Big(\frac{V}{1-e^{-2V}}\Big)_{Adj}\Big(e^{\boldsymbol{\Omega}}ce^{-\boldsymbol{\Omega}}\Big)\Big\};$$
$$S_{\mathsf{NK}} = \frac{1}{2e_0^2}\mathrm{tr}\int d^4x\, d^4\theta\, b^+\Big[e^{\boldsymbol{\Omega}^+}K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)e^{\boldsymbol{\Omega}}\Big]_{Adj}b.$$

The total action of the gauge fixed theory is invariant under the BRST transformations

$$\delta V = -\varepsilon\Big\{\Big(\frac{V}{1-e^{2V}}\Big)_{Adj}\Big(e^{-\boldsymbol{\Omega}^+}c^+e^{\boldsymbol{\Omega}^+}\Big) + \Big(\frac{V}{1-e^{-2V}}\Big)_{Adj}\big(e^{\boldsymbol{\Omega}}ce^{-\boldsymbol{\Omega}}\big)\Big\};$$

$$\delta\phi = \varepsilon c\phi; \qquad \delta\bar{c} = \varepsilon\bar{D}^2(e^{-2\boldsymbol{V}}f^+e^{2\boldsymbol{V}}); \qquad \delta\bar{c}^+ = \varepsilon D^2(e^{2\boldsymbol{V}}fe^{-2\boldsymbol{V}});$$
$$\delta c = \varepsilon c^2; \qquad \delta c^+ = \varepsilon(c^+)^2; \qquad \delta f = 0; \qquad \delta b = 0; \qquad \delta\boldsymbol{\Omega} = 0,$$

where ε is an anticommuting scalar parameter.

In our notation the renormalization constants are defined by the equations

$$\frac{1}{\alpha_0} = \frac{Z_\alpha}{\alpha}; \qquad \frac{1}{\xi_0} = \frac{Z_\xi}{\xi}; \qquad \boldsymbol{V} = \boldsymbol{V}_R; \qquad V = Z_V Z_\alpha^{-1/2} V_R;$$

$$b = \sqrt{Z_b} b_R; \qquad \bar{c}c = Z_c Z_\alpha^{-1} \bar{c}_R c_R; \qquad \phi_i = (\sqrt{Z_\phi})_i{}^j (\phi_R)_j;$$

$$m^{ij} = m_0^{mn} (Z_m)_m{}^i (Z_m)_n{}^j; \qquad \lambda^{ijk} = \lambda_0^{mnp} (Z_\lambda)_m{}^i (Z_\lambda)_n{}^j (Z_\lambda)_p{}^k.$$

The subscript R denotes renormalized superfields, α, λ, and ξ are the renormalized coupling constant, the Yukawa couplings, and the gauge parameter, respectively; m denotes renormalized masses.
It is possible to impose the following constrains to these renormalization constants:

$$(Z_m)_i{}^j = (Z_\lambda)_i{}^j = (\sqrt{Z_\phi})_i{}^j; \qquad Z_\xi = Z_V^{-2}; \qquad Z_b = Z_\alpha^{-1}.$$

Non-renormalization of the vertices with two ghost legs and one leg of the quantum gauge superfield

We will prove that the three-point vertices with two ghost legs and a single leg of the quantum gauge superfield are finite in all orders.

K.S., Nucl.Phys. **B909** (2016) 316.

There are 4 such vertices, $\bar{c}Vc$, $\bar{c}^+Vc$, $\bar{c}Vc^+$, and $\bar{c}^+Vc^+$.
They have the same renormalization constant $Z_\alpha^{-1/2}Z_cZ_V$. Therefore, the above statement can be rewritten as

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2}Z_cZ_V)=0.$$

In the one-loop approximation this has first been noted in the paper

S.S.Aleshin, A.E.Kazantsev, M.B.Skopsov, K.S., JHEP **1605** (2016) 014.

Consequently, there is a subtraction scheme in which

$$-\frac{1}{2}\ln Z_\alpha+\ln Z_c+\ln Z_V=0.$$

Important: Below we will demonstrate that Z_c is divergent. Therefore, The Green functions of the structure $\bar{c}V^nc$ are divergent for $n\neq 1$.

The Slavnov–Taylor identity can be derived by making the substitution coinciding with the BRST transformations in the generating functional and is written as

$$0 = \int d^4x\, d^4\theta_x\, \frac{\delta\Gamma}{\delta V_x^A} \left\langle \delta V_x^A \right\rangle + \int d^4x\, d^2\theta_x \Big(\left\langle \delta \bar{c}_x^A \right\rangle \frac{\delta\Gamma}{\delta \bar{c}_x^A} + \left\langle \delta c_x^A \right\rangle \frac{\delta\Gamma}{\delta c_x^A}$$
$$+ \left\langle \delta\phi_i \right\rangle \frac{\delta\Gamma}{\delta\phi_i} \Big) + \int d^4x\, d^2\bar{\theta}_x \Big(\left\langle \delta \bar{c}_x^{*A} \right\rangle \frac{\delta\Gamma}{\delta \bar{c}_x^{*A}} + \left\langle \delta c_x^{*A} \right\rangle \frac{\delta\Gamma}{\delta c_x^{*A}} + \left\langle \delta\phi^{*i} \right\rangle \frac{\delta\Gamma}{\delta\phi^{*i}} \Big),$$

where we keep the ε-dependence.
Also we will use the identity obtained by making the substitution $\bar{c} \to \bar{c} + a$, where a is an arbitrary chiral superfield:

$$\varepsilon \frac{\delta\Gamma}{\delta \bar{c}_x^A} = \frac{1}{4} \bar{D}^2 \left\langle \delta V_x^A \right\rangle; \qquad \varepsilon \frac{\delta\Gamma}{\delta \bar{c}_x^{*A}} = \frac{1}{4} D^2 \left\langle \delta V_x^A \right\rangle,$$

where, for simplicity, the background superfield is set to 0.

Let us differentiate the Slavnov–Taylor identity with respect to $\bar{c}_y^{*B}$, c_z^C, and c_w^D, set the fields to 0, and use the equations

$$\frac{\delta^2\Gamma}{\delta\bar{c}_y^{*B}\delta c_x^A} = -\frac{D_y^2\bar{D}_x^2}{16}G_c\delta_{xy}^8\delta_{AB}; \qquad \frac{\delta}{\delta c_x^A}\left\langle\delta V_y^B\right\rangle = -\varepsilon\cdot\frac{1}{4}G_c\,\bar{D}^2\delta_{xy}^8\delta_{AB}.$$

As a result we obtain the identity

$$\varepsilon\cdot G_c(\partial_w^2/\Lambda^2)\bar{D}_w^2\frac{\delta^3\Gamma}{\delta\bar{c}_y^{*B}\delta V_w^D\delta c_z^C} - \varepsilon\cdot G_c(\partial_z^2/\Lambda^2)\bar{D}_z^2\frac{\delta^3\Gamma}{\delta\bar{c}_y^{*B}\delta V_z^C\delta c_w^D}$$
$$+\frac{1}{2}G_c\left(\partial_y^2/\Lambda^2\right)D_y^2\frac{\delta^2}{\delta c_z^C\delta c_w^D}\left\langle\delta c_y^B\right\rangle = 0.$$

Similarly, differentiating with respect to $\bar{c}_y^{*B}$, c_z^{*C}, and c_w^D gives

$$\varepsilon\cdot G_c(\partial_w^2/\Lambda^2)\bar{D}_w^2\frac{\delta^3\Gamma}{\delta\bar{c}_y^{*B}\delta V_w^D\delta c_z^{*C}} + \varepsilon\cdot G_c(\partial_z^2/\Lambda^2)D_z^2\frac{\delta^3\Gamma}{\delta\bar{c}_y^{*B}\delta V_z^C\delta c_w^D}$$
$$+\frac{1}{2}G_c\left(\partial_y^2/\Lambda^2\right)D_y^2\frac{\delta^2}{\delta c_z^{*C}\delta c_w^D}\left\langle\delta c_y^B\right\rangle = 0.$$

Explicit expressions for the three-point ghost-gauge functions

To simplify these identities we use explicit expressions for the Green functions. They can be derived using dimensional and chirality considerations:

$$\frac{\delta^3\Gamma}{\delta\bar{c}^{*A}_x\delta V^B_y\delta c^C_z} = -\frac{ie_0}{16}f^{ABC}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\Big(f(p,q)\partial^2\Pi_{1/2}$$
$$-F_\mu(p,q)(\gamma^\mu)_{\dot{a}}{}^b\bar{D}^{\dot{a}}D_b + F(p,q)\Big)_y\Big(D^2_x\delta^8_{xy}(q+p)\,\bar{D}^2_z\delta^8_{yz}(q)\Big);$$

$$\frac{\delta^3\Gamma}{\delta\bar{c}^{*A}_x\delta V^B_y\delta c^{*C}_z} = -\frac{ie_0}{16}f^{ABC}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\widetilde{F}(p,q)D^2_x\delta^8_{xy}(q+p)D^2_z\delta^8_{yz}(q),$$

where $\partial^2\Pi_{1/2} \equiv -D^a\bar{D}^2D_a/8$ is the supersymmetric projection operator, and

$$\delta^8_{xy}(p) \equiv \delta^4(\theta_x-\theta_y)e^{ip_\alpha(x^\alpha-y^\alpha)}.$$

This implies that $q+p$ is the momentum of $\bar{c}^*$, $-p$ is the momentum of V, and $-q$ is the momentum of c (or c^*).

Explicit expressions for the ghost correlators

Let us introduce the chiral source $\mathcal{J}$ and add the term

$$-\frac{e_0}{2}\int d^4x\, d^2\theta\, f^{ABC}\mathcal{J}^A c^B c^C + \text{c.c.}$$

to the action. From dimensional and chirality considerations we obtain

$$\frac{\delta^2}{\delta c_z^C \delta c_w^D}\left\langle \delta c_y^B \right\rangle = -i\varepsilon \cdot \frac{\delta^3\Gamma}{\delta c_z^C \delta c_w^D \delta \mathcal{J}_y^B}$$

$$= -\frac{ie_0\varepsilon}{4} f^{BCD}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} H(p,q)\bar{D}_z^2\delta_{zy}^8(q+p)\bar{D}_w^2\delta_{yw}^8(q);$$

$$\frac{\delta^2}{\delta c_z^{*C} \delta c_w^D}\left\langle \delta c_y^B \right\rangle = -i\varepsilon \cdot \frac{\delta^3\Gamma}{\delta c_z^{*C} \delta c_w^D \delta \mathcal{J}_y^B}$$

$$= -\frac{ie_0\varepsilon}{64} f^{BCD}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \widetilde{H}(p,q)\bar{D}_y^2 D_y^2\Big(D_z^2\delta_{zy}^8(q+p)\bar{D}_w^2\delta_{yw}^8(q)\Big).$$

where $[H(p,q)] = 1$, $[\widetilde{H}(p,q)] = m^{-2}$, and, by construction,

$$H(p,q) = H(p,-q-p).$$

Substituting explicit expressions for the Green functions into the Slavnov–Taylor identities, we can rewrite them in the form

$$G_c(q)F(q,p) + G_c(p)F(p,q) = 2G_c(q+p)H(-q-p,q);$$

$$G_c(q)\widetilde{F}(q,p) - G_c(p)\Big(F(p,q) - 4p^\mu F_\mu(p,q)\Big)$$

$$= 2G_c(q+p)(q+p)^2\widetilde{H}(-q-p,q).$$

For simplicity, here we use the notation $G_c(-q^2/\Lambda^2) \to G_c(q)$. The scalar products are constructed by the help of the Minkowski metric with the signature $(+---)$.
The first identity will be used below for proving finiteness of the ghost-gauge vertices.

Finiteness of the function H

First, let us prove that the function $H(p,q)$ is finite. H is contributed by diagrams in which one leg corresponds to the chiral source $\mathcal{J}$ and two other legs correspond to chiral ghost superfields c. These diagrams contain

$$\int d^4y\, d^2\theta_y\, \mathcal{J}^A_y \cdot \frac{\bar{D}^2_y D^2_y}{4\partial^2}\delta^8_{y1} \cdot \frac{\bar{D}^2_y D^2_y}{4\partial^2}\delta^8_{y2} = -2\int d^4y\, d^4\theta_y\, \mathcal{J}^A_y \cdot \frac{D^2_y}{4\partial^2}\delta^8_{y1} \cdot \frac{\bar{D}^2_y D^2_y}{4\partial^2}\delta^8_{y2}.$$

Therefore, the considered contribution can be presented as an integral over the total superspace, which includes integration over

$$\int d^4\theta = -\frac{1}{2}\int d^2\theta \bar{D}^2 + \text{ total derivatives in the coordinate space }.$$

This implies that two left spinor derivatives should act to the chiral external lines. Therefore, the non-vanishing result can be obtained only if two right spinor derivatives also act to the external lines. Consequently, the result should be proportional to, at least, second degree of the external momenta and is finite in the ultraviolet region.
Thus, the function $H(p,q)$ is UV finite.

Non-renormalization of the three-point ghost-gauge vertices

Let us multiply the Slavnov–Taylor identity to the renormalization constant Z_c (such that $(G_c)_R = Z_c G$ is finite), differentiate the result with respect to $\ln\Lambda$, and take the limit $\Lambda \to \infty$. Due to finiteness of $(G_c)_R$ and H the result is written as

$$\Big((G_c)_R(q)\frac{d}{d\ln\Lambda}F(q,p) + (G_c)_R(p)\frac{d}{d\ln\Lambda}F(p,q)\Big)\Big|_{\Lambda\to\infty} = 0.$$

Setting $p = -q$, we obtain

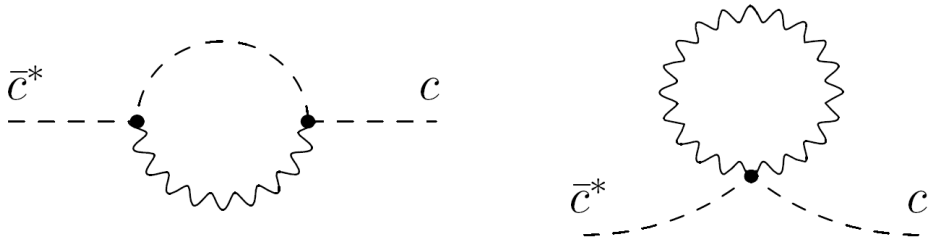
$$\frac{d}{d\ln\Lambda}F(-q,q)\Big|_{\Lambda\to\infty} = 0.$$

Therefore, the corresponding renormalization constant is finite

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2} Z_c Z_V) = 0.$$

Thus, the function $F(p,q)$ is also finite. This implies that all three-point ghost-gauge vertices are finite.

One-loop calculation: two-point ghost Green function



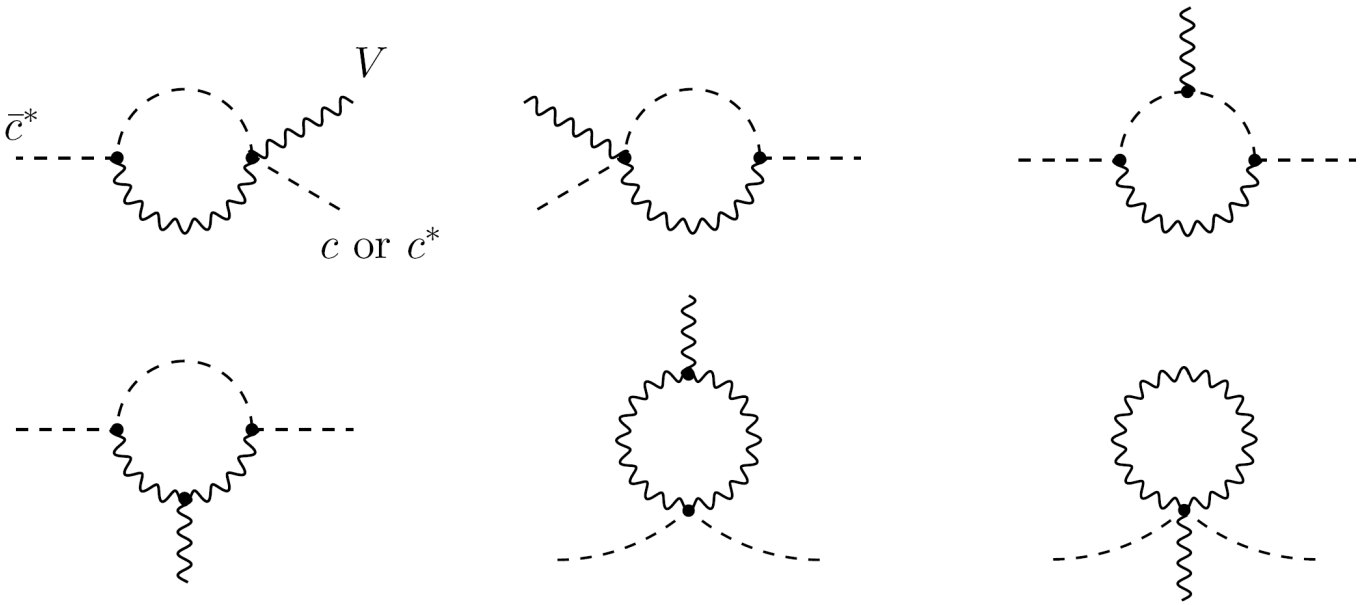
In the Euclidean space after the Wick rotation

$$G_c(p) = 1 + e_0^2 C_2 \int \frac{d^4k}{(2\pi)^4} \Big(\frac{\xi_0}{K_k} - \frac{1}{R_k} \Big) \Big(-\frac{1}{6k^4} + \frac{1}{2k^2(k+p)^2} - \frac{p^2}{2k^4(k+p)^2} \Big) + O(e_0^4, e_0^2\lambda_0^2),$$

where $R_k \equiv R(k^2/\Lambda)$ and $K_k \equiv K(k^2/\Lambda^2)$.
We see that this function is divergent in the ultraviolet region (at infinite Λ).

$$\gamma_c(\alpha_0, \lambda_0) = \frac{d \ln G_c}{d \ln \Lambda}\Big|_{p=0;\, \alpha,\lambda=\text{const}} = -\frac{\alpha_0 C_2 (1-\xi_0)}{6\pi} + O(\alpha_0^2, \alpha_0\lambda_0^2).$$

One-loop calculation: three-point gauge-ghost Green functions



$$\frac{ie_0}{4} f^{ABC} \int d^4\theta \, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \Big(f(p,q) \partial^2 \Pi_{1/2} V^B(\theta, -p)$$

$$+ F_\mu(p,q) (\gamma^\mu)_{\dot{a}}{}^b D_b \bar{D}^{\dot{a}} V^B(\theta, -p) + F(p,q) V^B(\theta, -p) \Big) c^C(\theta, -q);$$

$$\frac{ie_0}{4} f^{ABC} \int d^4\theta \, \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{c}^{*A}(\theta, p+q) \widetilde{F}(p,q) V^B(\theta, -p) c^{*C}(\theta, -q).$$

Calculating these diagrams gives

$$F(p,q) = 1 + \frac{e_0^2 C_2}{4} \int \frac{d^4k}{(2\pi)^4} \Big\{ - \frac{(q+p)^2}{R_k k^2 (k+p)^2 (k-q)^2} - \frac{\xi_0\, p^2}{K_k k^2 (k+q)^2 (k+q+p)^2}$$
$$+ \frac{\xi_0\, q^2}{K_k k^2 (k+p)^2 (k+q+p)^2} + \left(\frac{\xi_0}{K_k} - \frac{1}{R_k} \right) \left(- \frac{2(q+p)^2}{k^4 (k+q+p)^2} + \frac{2}{k^2 (k+q+p)^2}$$
$$- \frac{1}{k^2 (k+q)^2} - \frac{1}{k^2 (k+p)^2} \right) \Big\} + O(\alpha_0^2, \alpha_0 \lambda_0^2).$$

$$\widetilde{F}(p,q) = 1 - \frac{e_0^2 C_2}{4} \int \frac{d^4k}{(2\pi)^4} \Big\{ \frac{p^2}{R_k k^2 (k+q)^2 (k+q+p)^2} + \frac{\xi_0\, (q+p)^2}{K_k k^2 (k-p)^2 (k+q)^2}$$
$$+ \frac{\xi_0\, q^2}{K_k k^2 (k+p)^2 (k+q+p)^2} + \frac{2\xi_0}{K_k k^2 (k+p)^2} - \frac{2\xi_0}{K_k k^2 (k+q+p)^2} + \left(\frac{\xi_0}{K_k} - \frac{1}{R_k} \right)$$
$$\times \left(\frac{2q^2}{k^4 (k+q)^2} + \frac{1}{k^2 (k+q+p)^2} - \frac{1}{k^2 (k+q)^2} \right) \Big\} + O(\alpha_0^2, \alpha_0 \lambda_0^2).$$

We see that these expressions are finite in the ultraviolet region.

One-loop calculation: the function f

The expressions for the functions f and F_μ are very large and in writing them we will use the notation

$$\Delta_q \equiv \frac{\xi_0}{K_q} - \frac{1}{R_q}.$$

The function f has the form

$$f(p,q) = \frac{1}{4}\int \frac{d^4k}{(2\pi)^4}\frac{e_0^2 C_2}{k^2(k+q)^2(k+q+p)^2}\Big\{\frac{2k_\mu q_\mu}{(k+q)^2}\Delta_{k+q} + \frac{2k^2}{(k+q+p)^2}\Delta_{k+q+p}$$
$$+R_p\Big(\frac{2k_\mu(q+p)^\mu}{(k+q+p)^2 R_{k+q}}\Delta_{k+q+p} + \frac{2k^2}{(k+q)^2 R_{k+q+p}}\Delta_{k+q} + \Big(\frac{k_\mu(k+q+p)^\mu}{(k+q+p)^2}$$
$$+\frac{k_\mu(k+q)^\mu}{(k+q)^2}\Big)\Delta_{k+q}\Delta_{k+q+p}\Big) - \frac{2k_\mu(k+q)^\mu}{R_{k+q}R_{k+q+p}}\cdot\frac{R_{k+q+p}-R_{k+q}}{(k+q+p)^2-(k+q)^2}$$
$$-\frac{2(R_{k+q+p}-R_p)}{(k+q+p)^2-p^2}\cdot\frac{1}{R_{k+q+p}}\Big(\frac{k_\mu q^\mu(k+q+p)^2 - k_\mu q^\mu p^2}{(k+q)^2}\Delta_{k+q} + \frac{k_\mu p^\mu}{R_{k+q}}\Big)$$
$$-\frac{2(R_{k+q}-R_p)}{(k+q)^2-p^2}\cdot\frac{1}{R_{k+q}}\Big(\frac{k^2(k+q)^2 - k^2p^2}{(k+q+p)^2}\Delta_{k+q+p} + \frac{k_\mu(k+q)^\mu}{R_{k+q+p}}\Big)\Big\} + O(e_0^4, e_0^2\lambda_0^2).$$

One-loop calculation: the function F_μ

$$F_\mu(p,q) = \frac{1}{16}\int \frac{d^4k}{(2\pi)^4}\frac{e_0^2 C_2}{k^2(k+q)^2(k+q+p)^2}\Big\{\frac{2}{k^2}\Delta_k\Big[(q+p)_\mu\, k_\alpha (k+q)^\alpha + q_\mu\, k_\alpha$$

$$\times(k+q+p)^\alpha + k_\mu\Big(k^2 - q^2 - q_\alpha p^\alpha\Big)\Big] - \frac{4k_\mu}{R_{k+q}} + \frac{2}{(k+q)^2}\Delta_{k+q}\Big[-q_\mu k_\alpha p^\alpha + p_\mu k^2$$

$$+k_\mu q_\alpha p^\alpha - k_\mu (k+q)^2 + k_\alpha q^\alpha (2q+2k+p)_\mu\Big] + \frac{2}{(k+q+p)^2}\Delta_{k+q+p}\Big[q_\mu k_\alpha (q+p)^\alpha$$

$$+(q+p)_\mu k_\alpha q^\alpha - k_\mu(q^2 + q_\alpha p^\alpha + k^2) - p_\mu k^2\Big] - \frac{R_{k+q+p} - R_{k+q}}{(k+q+p)^2 - (k+q)^2}(2q+2k+p)_\mu$$

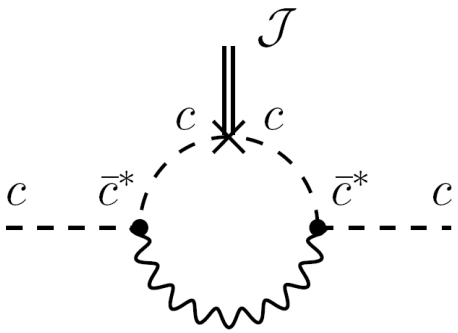
$$\times\frac{4k^\alpha q_\alpha}{R_{k+q}R_{k+q+p}} + \frac{2R_p}{(k+q)^2(k+q+p)^2}\Delta_{k+q+p}\Delta_{k+q}\Big[(p_\mu p^\nu - \delta_\mu^\nu p^2)\Big((k^2+q^2)(k_\nu + q_\nu)$$

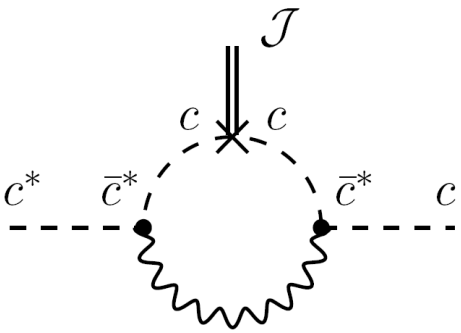
$$-(k+q)^2 q_\nu\Big) + p^2(q_\mu k_\alpha p^\alpha - k_\mu q_\alpha p^\alpha)\Big] + \frac{4R_p}{(k+q)^2 R_{k+q+p}}\Delta_{k+q}\,(q_\mu k_\alpha p^\alpha - k_\mu q_\alpha p^\alpha)$$

$$+\frac{4(R_{k+q} - R_p)}{(k+q)^2 - p^2}\frac{(k_\mu q_\alpha p^\alpha - q_\mu k_\alpha p^\alpha)}{R_{k+q}R_{k+q+p}} + \frac{4(R_{k+q+p} - R_p)}{(k+q+p)^2 - p^2}\Big(\frac{(p_\mu p^\nu - \delta_\mu^\nu p^2)k_\nu}{R_{k+q+p}R_{k+q}} + \Delta_{k+q}$$

$$\times\frac{\big((k+q+p)^2 - p^2\big)}{(k+q)^2 R_{k+q+p}}\Big(q_\mu k_\alpha p^\alpha - k_\mu q_\alpha p^\alpha\Big)\Big)\Big\} + O(e_0^4, e_0^2\lambda_0^2).$$

One-loop calculation: finiteness of the function H





$$H(p,q) = 1 - \frac{e_0^2 C_2}{4} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{p^2}{R_k k^2 (k+q)^2 (k+q+p)^2} \right.$$

$$\left. + \frac{(q+p)^2}{k^4 (k+q+p)^2} \left(\frac{\xi_0}{K_k} - \frac{1}{R_k} \right) + \frac{q^2}{k^4 (k+q)^2} \left(\frac{\xi_0}{K_k} - \frac{1}{R_k} \right) \right\} + O(e_0^4, e_0^2 \lambda_0^2);$$

$$\widetilde{H}(p,q) = \frac{e_0^2 C_2}{4} \int \frac{d^4k}{(2\pi)^4} \frac{1}{K_k k^2 (k+q)^2 (k+q+p)^2} + O(e_0^4, e_0^2 \lambda_0^2).$$

We see that the function H is finite in the ultraviolet region and is quadratic in external momenta.

One can verify that the above functions satisfy the Slavnov–Taylor identities

$$G_c(-q-p)H(-q-p,q) = 1 + \frac{e_0^2 C_2}{4}\int \frac{d^4k}{(2\pi)^4}\Big\{ -\frac{(q+p)^2}{R_k k^2 (k+p)^2 (k-q)^2}$$

$$+\Big(\frac{\xi_0}{K_k} - \frac{1}{R_k}\Big)\Big(\frac{2}{k^2(k+q+p)^2} - \frac{2(q+p)^2}{k^4(k+q+p)^2} - \frac{p^2}{k^4(k+p)^2} - \frac{q^2}{k^4(k-q)^2}$$

$$-\frac{2}{3k^4}\Big)\Big\} + O(\alpha_0^2, \alpha_0\lambda_0^2) = \frac{1}{2}\Big(G_c(q)F(q,p) + G_c(p)F(p,q)\Big).$$

$$G_c(q)\widetilde{F}(q,p) - G_c(p)\Big(F(p,q) + 4p^\mu F_\mu(p,q)\Big)$$

$$= -\frac{e_0^2 C_2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{(q+p)^2}{K_k k^2 (k-p)^2 (k+q)^2} + O(\alpha_0^2, \alpha_0\lambda_0^2)$$

$$= -2G_c(q+p)(q+p)^2\widetilde{H}(-q-p,q).$$

Note that earlier we use Minkowski momenta, while here the momenta are Euclidian. Therefore, due to the identity $(a_\mu b^\mu)_M = -(a_\mu b^\mu)_E$ some signs are different.

NSVZ β-function

In $\mathcal{N} = 1$ supersymmetric theories the β-function is related to the anomalous dimension of the matter superfields by the equation

$$\beta(\alpha) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j\gamma_j{}^i(\alpha)/r\Big)}{2\pi(1 - C_2\alpha/2\pi)}.$$

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. B 229, (1983), 381; Phys.Lett. 166B, (1985), 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. B 277, (1986), 456.

The NSVZ β-function have been derived from various general arguments: instantons, anomalies, etc.
For $\mathcal{N} = 1$ SQED, regularized by higher derivatives, the NSVZ relation has been obtained by explicit summation of supergraphs

K.S., Nucl.Phys. **B 852** (2011) 71; JHEP **1408** (2014) 096.

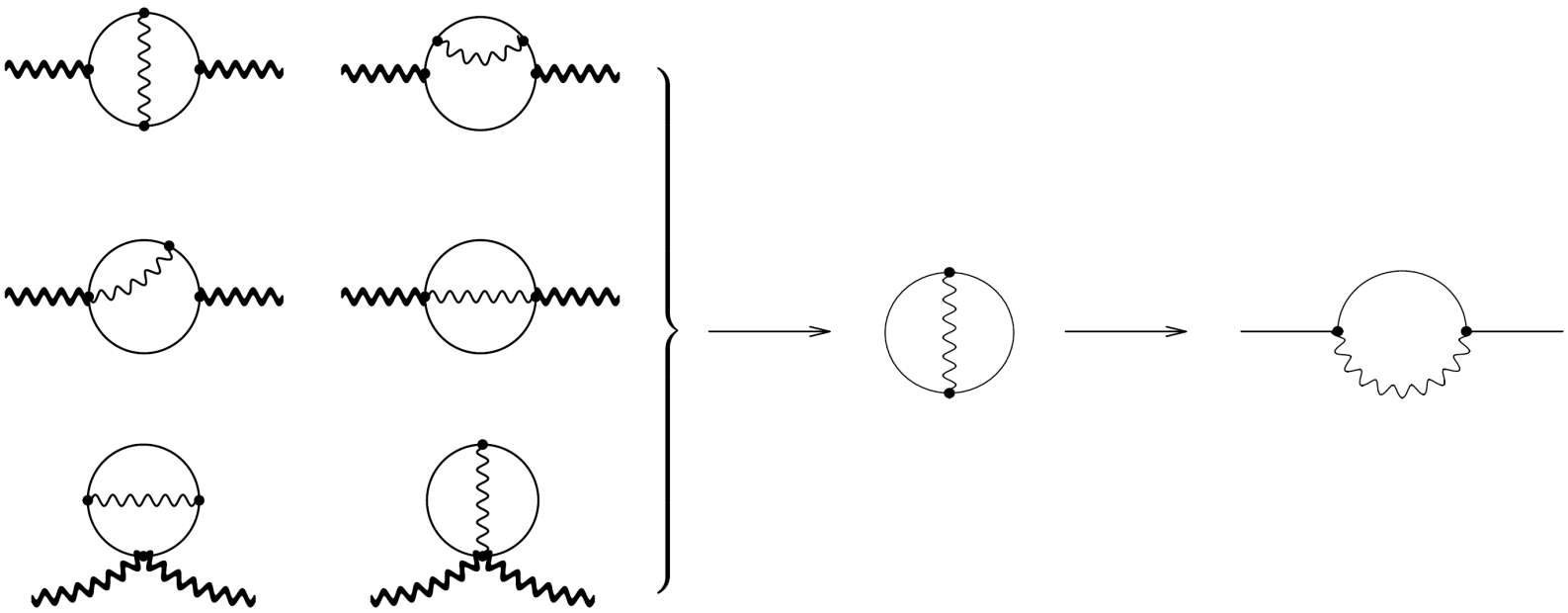
Generalization of this result to the case of using the dimensional reduction is a complicated and so far unsolved problem

S.S.Aleshin, A.L.Kataev, K.S., JETP Lett. **103** (2016) 77.

Derivation of the NSVZ β-function in the Abelian case by summing supergraphs

Qualitative picture:

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445.



Renormalization group functions defined in terms of the bare coupling constants

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda} = \alpha_0^2 \frac{d}{d\ln\Lambda}\Big(d^{-1}(\alpha_0,\lambda_0,\Lambda/p) - \alpha_0^{-1}\Big)\Big|_{p=0};$$

$$(\gamma_\phi)_i{}^j(\alpha_0,\lambda_0) \equiv -\frac{d\ln(Z_\phi)_i{}^j(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda} = \frac{d\ln(G_\phi)_i{}^j(\alpha_0,\lambda_0,\Lambda/p)}{d\ln\Lambda}\Big|_{p=0};$$

$$\gamma_V(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_V(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda} = \frac{1}{2}\cdot\frac{d\ln G_V(\alpha_0,\lambda_0,\Lambda/p)}{d\ln\Lambda}\Big|_{p=0};$$

$$\gamma_c(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_c(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda} = \frac{d\ln G_c(\alpha_0,\lambda_0,\Lambda/p)}{d\ln\Lambda}\Big|_{p=0}.$$

where the differentiation is made at fixed values of α and λ^{ijk}.
There renormalization group functions are
1. scheme independent at a fixed regularization;
2. depend on a regularization;
3. satisfy the NSVZ relation in all orders for $\mathcal{N}=1$ SQED with N_f flavors, regularized by higher derivatives.

Renormalization group functions defined in terms of the renormalized coupling constant

The renormalization group functions are standardly defined in terms of the renormalized coupling constants by the equations

$$\widetilde{\beta}(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu};$$

$$(\widetilde{\gamma}_\phi)_i{}^j(\alpha,\lambda) \equiv \frac{d\ln(Z_\phi)_i{}^j(\alpha,\lambda,\Lambda/\mu)}{d\ln\mu};$$

$$\widetilde{\gamma}_V(\alpha,\lambda) \equiv \frac{d\ln Z_V(\alpha,\lambda,\Lambda/\mu)}{d\ln\mu};$$

$$\widetilde{\gamma}_c(\alpha,\lambda) \equiv \frac{d\ln Z_c(\alpha,\lambda,\Lambda/\mu)}{d\ln\mu}.$$

where the differentiation is made at fixed values of α_0 and λ_0^{ijk}.
There renormalization group functions are
1. scheme dependent;
2. satisfy the NSVZ relation only in a special subtraction scheme, which is called the NSVZ scheme.

New form of the NSVZ β-function

The NSVZ β-function can be equivalently rewritten in the form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

Let us express the β-function in the right hand side in terms of the renormalization constant Z_α:

$$\beta(\alpha_0,\lambda_0) = \frac{d\alpha_0(\alpha,\lambda,\Lambda/\mu)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\alpha_0\frac{d\ln Z_\alpha}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}.$$

Then, using the identity $d(Z_\alpha^{-1/2}Z_V Z_c)/d\ln\Lambda = 0$ we obtain

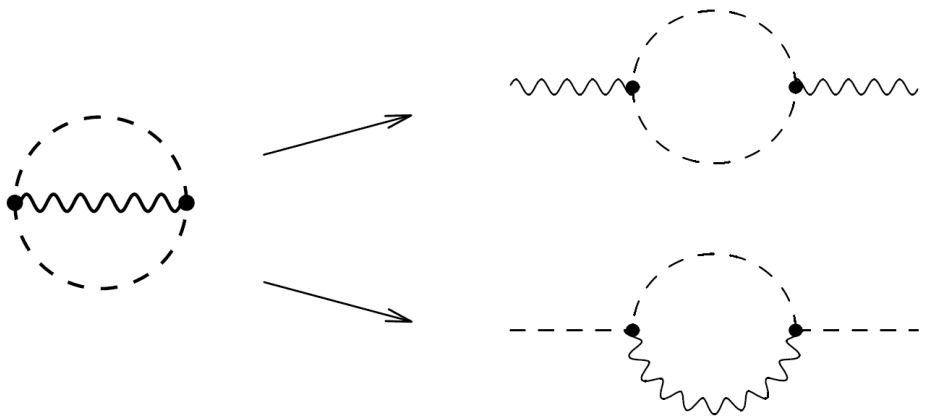
$$\beta(\alpha_0,\lambda_0) = -2\alpha_0\frac{d\ln(Z_c Z_V)}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0)+\gamma_V(\alpha_0,\lambda_0)\Big),$$

where γ_c and γ_V are anomalous dimensions of the Faddeev–Popov ghosts and of the quantum gauge superfield (defined in terms of the bare coupling constants), respectively.

Substituting this expression into the right hand side of the NSVZ relation we obtain

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

From this form of the NSVZ β-function we see that the matter superfields and ghosts similarly contribute to the right hand side.



Let us assume that this equation is valid with the higher derivative regularization similarly to the Abelian case.

The NSVZ scheme in the non-Abelian case

The RG functions defined in terms of the renormalized coupling constant are scheme dependent and satisfy the NSVZ relation only in a certain subtraction scheme. Similarly to

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459; Phys.Lett. **B730** (2014) 184.

we see that in the non-Abelian case the RG functions defined in terms of the bare coupling constant coincide with ones defined in terms of the renormalized coupling constants if the boundary conditions

$$Z_\alpha(\alpha,\lambda,x_0)=1; \qquad (Z_\phi)_i{}^j(\alpha,\lambda,x_0)=\delta_i{}^j; \qquad Z_c(\alpha,\lambda,x_0)=1,$$

where x_0 is a fixed value of $\ln\Lambda/\mu$, are imposed on the renormalization constants. (For example, it is possible and convenient to choose $x_0=0$.) We also assume that the renormalization constants satisfy the equation

$$Z_V = Z_\alpha^{1/2} Z_c^{-1},$$

Possibly, these conditions give the NSVZ scheme with the higher covariant derivative regularization.

Conclusion

- ✓ For $\mathcal{N}=1$ SYM the three-point vertices with two ghost legs and a single leg of the quantum gauge superfield are finite. This has been proved in all orders using the Slavnov–Taylor identities and has been verified by an explicit one-loop calculation.
- ✓ Due to non-renormalization of the three-point ghost-gauge vertices the renormalization constants can be chosen so that $Z_\alpha^{-1/2} Z_V Z_c = 1$.
- ✓ The NSVZ β-function can be rewritten in terms of the anomalous dimensions of the quantum gauge superfield, of the Faddeev–Popov ghosts, and of the matter superfields. The resulting expression for the NSVZ β-function has a simple qualitative interpretation.
- ✓ Using the above results it is possible to suggest a simple prescription giving the NSVZ scheme in the non-Abelian case, if the higher covariant derivative method is used for regularization.

Thank you for the attention!