Comments on conformal higher spins in curved backgrounds

Maxim Grigoriev

Lebedev Physical Institute, Moscow

Based on: *M.G., A. Tseytlin – arXiv:1609.09381, to appear in JPA*

HSTH-5

Lebedev Physical Institute, Moscow
Dec 1, 2016

Conformal higher spin (CHS) fields

Fradkin Tseytlin 1985

Tseytlin 2002, Segal 2002

$s > 2$ generalizations of Maxwell $(F_{\mu\nu})^2$ and Weyl $(C_{\mu\nu\lambda\rho})^2$ theories

The free CHS action in flat 4-dimensional space may be written as

$$S_s = \int d^4x \, h_s P_s \, \partial^{2s} \, h_s = \int d^4x \, (-1)^s \, C_s C_s$$

where $h_s = (h_{\mu_1 \ldots \mu_s})$ is a totally symmetric tensor.
Gauge symmetries:

$$\delta h_{\mu_1 \ldots \mu_s} = \partial_{(\mu_1} \epsilon_{\mu_2 \ldots \mu_{s-1})} + \eta_{(\mu_1 \mu_2} \omega_{\mu_3 \ldots \mu_s)}$$

In flat space this action is invariant under the conformal (i.e. $o(4,2)$) symmetries.

Thanks to the conformal invariance CHS fields propagate consistently on any conformally-flat background. Explicit description can e.g. be obtained from the ambient-space formulation

Bekaert, Grigoriev, 2012

In contrast, Fronsdal HS fields are quite rigid and require constant curvature background.

A natural question is whether CHS fields are consistent over a not necessarily conformally-flat geometry?

To first order in curvatures conformal spin 3 is consistent on Bach flat geometry *Nutma, Taronna 2014*

Our approach employs the existence of a full nonlinear CHS theory

Segal 2002

earlier proposal: *Tseytlin, 2002*

Quantized particle in the background fields

Segal 2002

Also: *MG 2006, Bekaert, Joung, Mourad 2010*

Particle phase space x^μ, p_ν. *-product of (Weyl) symbols:

$$* = \exp\left[\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\partial}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p_\mu}\frac{\partial}{\partial x^\mu}\right)\right].$$

Particle is described by the only 1st class constraint (Hamiltonian)

$$H(x,p)\,, \qquad H(x,p) = \sum_{s=0}^{\infty} H^{\mu_1\cdots\mu_s}(x)\, p_{\mu_1}\ldots p_{\mu_s}$$

interpreted as a generating function of fields $H^{\mu_1\cdots\mu_s}(x)$.

Natural gauge transformations: *Segal, 2002*

$$\delta H = [H, \epsilon(x,p)]_\star + \{H, \omega(x,p)\}_*$$

generalised gradient transf. (ϵ) + generalised Weyl transf. (ω)

These gauge symmetries are natural symmetries of the constrained system. More generally, for a system

$$T_a(x,p) = 0, \qquad [T_a, T_b]_\star = U_{ab}^c * T_c\,.$$

an infinitesimal canonical transformation

$$T_a \sim T_a + [T_a, \epsilon]_\star$$

is a natural gauge equivalence. The equivalence relation

$$T_a \sim T_a + \omega_a^b * T_b$$

corresponds to an infinitesimal redefinition of the constraints (which preserve the constraint surface).

The space of gauge-inequivalent configurations is a moduli space of constrained systems that have fixed number of 1-st class constraints and satisfy certain extra conditions (e.g. they belong to a vicinity of certain vacuum system).
Note that in the case $\{T_a\} = H$, the equations are trivial while the gauge symmetries are just the ones of *Segal, 2002*

In the BRST formalism the constraints are encoded in the symbol $\Omega(x, p, c, \mathcal{P})$ of the BRST operator.

The equations and gauge transformations are encoded in

$$[\Omega, \Omega]_\star = 0\,, \qquad \delta\Omega = [\Omega, \Xi]_\star$$

$$\Omega = c^a T_a(x, p) + \dots\,, \qquad \Xi = \epsilon + c^a \omega_a^b \mathcal{P}_b$$

In the context of String Field Theory: *Horowitz et all, 1986*.

Representation space: functions in x^μ. Assume $\widehat{H}, \widehat{\omega}, \imath\widehat{\epsilon}$ formally hermitean, the action of the scalar field on the background H:

Segal 2002

$$S[\phi, H] = \int d^d x \; \phi^*(x) \; \widehat{H}(x, \frac{\partial}{\partial x}) \; \phi(x)$$

Invariant under the gauge transformations for h and ϕ:

$$\delta\phi = -(\widehat{\epsilon} + \widehat{\omega})\phi \,, \qquad \delta H = [H, \epsilon(x,p)]_\star + \{H, \omega(x,p)\}_*$$

Conserved currents

Take $H = H_0 + h$,

$$S[\phi, h] = \int d^d x \left[\phi^* \widehat{H}_0 \phi + \phi^* \widehat{h} \phi\right] .$$

For $\omega = 0$ the gauge invariance of S can be written as

$$\int d^d x \left(([H_0, \epsilon]_* + [h, \epsilon]) \frac{\delta S}{\delta h} - \widehat{\epsilon}\, \phi^* \frac{\delta S}{\delta \phi} \right) = 0 .$$

Introducing currents $J_{\mu_1 \dots \mu_s}$

$$J_{\mu_1 \dots \mu_s}[\phi] = \frac{\delta S}{\delta h^{\mu_1 \dots \mu_s}} , \qquad S[h, \phi] = \int \phi^* \widehat{H}_0 \phi + h^{\mu_1 \cdots \mu_s} J_{\mu_1 \dots \mu_s}$$

The h-independent contribution gives

$$\int d^d x \left(\langle [H_0, \epsilon]_\star, J \rangle - 2(\widehat{\epsilon}\phi)^* \widehat{H}_0 \phi \right) = 0 ,$$

which for $H_0 = -\frac{1}{2}\eta^{\mu\nu} p_\mu p_\nu$ gives

$$\partial^{\mu_1} J_{\mu_1 \dots \mu_s} = 0 \quad \text{provided} \quad \Box \phi = 0$$

Analogously, gauge invariance with $\epsilon = 0$ gives "deformed" tracelessness condition

$$(\eta^{\mu_1\mu_2} J_{\mu_1\dots\mu_s} - \frac{1}{2}\Box) J_{\mu_3\dots\mu_s} = 0 \quad \text{provided} \quad \Box\phi = 0$$

After redefinition these currents provide a different parameterization

Didenko, Skvortsov 2012

of the usual Noether currents associated to the symmetries of the scalar field ϕ on the Minkowski space *Eastwood, 2002*

Their algebra is the familiar higher spin algebra of AdS_{d+1} HS fields or conformal HS fields in d dimensions *Vasiliev, Fradkin-Linetsky*

CHS action

Expanding around a background,

$$H = H_0 + h$$

the action for h can be obtained (at least in principle) by integrating over the scalar field and extracting the log-divergent piece $S_{CHS}[H]$ of the resulting effective action:

Tseytlin 2002, Segal 2002

Bekaert, Joung, Mourad 2010

$$\int D\phi \, e^{S[\phi,h]}$$

S_{CHS} is known to inherit the invariances of $S[\phi, h]$. In particular, it is invariant under the gauge symmetries:

$$\delta h = [H, \epsilon(x,p)]_\star + \{H, \omega(x,p)\}_*$$

For $H_0 = -\frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu$ one has

$$\delta h = p^\mu \partial_\mu \epsilon - (\eta^{\mu\nu}p_\mu p_\nu)\,\omega - \frac{1}{2}\Box\omega \; + \text{nonlinear terms}$$

i.e. is a deformation of the usual gauge symmetries of CHS.

The construction is only tractable for H_0 with $g^{\mu\nu} = \eta^{\mu\nu}$.

In particular:

For $g^{\mu\nu} = \eta^{\mu\nu}$ $H = H_0$ is a vacuum solution of CHS theory. In general this is not clear.

The analysys of more general H_0 is complicated becuase the construction lacks manifest covariance. Non-tensors.

A natural question is which H_0 are vacuum solutions? In other words, which backgrounds admit consistent propagation of linear CHS fields.

Of course, conformally-flat $g^{\mu\nu}$ are vacuum solutions.

Geometrically covariant approach

– We are used to work with tensor fields.
– It is desirable to have a geometrically-covariant framework for the background fields.
– Can be achieved by employing the covariant quantization technique combined with a version of background field method.

Fedosov-like quantization on cotangent bundles

Fedosov 1994, Bordemann et all, 1997

$T^*\mathcal{X}$ with coordinates x^μ, p_ν
Covariant quantization: map from functions in (x,p) (symbols) to differential operators (quantum operators) on functions in x (quantum states) defined in a coordinate-independent way. In particular, this determines a $*$-product of symbols:

$$f \star g = \widehat{\widehat{f}\widehat{g}}$$

The construction:

Pick a torsion-free connection $\Gamma^{\rho}_{\mu\nu}$
(in what follows, the one compatible with a given metric $g_{\mu\nu}$)
Extended space $T\mathcal{X} \oplus T^*\mathcal{X}$ with coordinates $x^{\mu}, p_{\nu}, y^{\rho}$
It is equipped with the $\circ$-product

$$\circ = exp\Big[\frac{1}{2}\big(\frac{\overleftarrow{\partial}}{\partial y^{\mu}}\frac{\partial}{\partial p_{\mu}} - \frac{\overleftarrow{\partial}}{\partial p_{\mu}}\frac{\partial}{\partial y^{\mu}}\big)\Big]\,.$$

$\Gamma^{\rho}_{\mu\nu}$ can be lifted to a nonlinear flat connection:

$$D = \nabla - dx^{a}\frac{\partial}{\partial y^{a}} + [\mathbf{r}, \cdot]_{\circ}\,, \qquad D^2 = 0$$

$$\mathbf{r} = dx^{\mu}\big[\frac{1}{3}R_{\mu\nu}{}^{\rho}{}_{\sigma}p_{\rho}y^{\nu}y^{\sigma} + \nabla Rpyyy + \ldots + RRpyyyy + \ldots\big]$$

For any $f(x,p)$ the connection determines a unique lift $F(x,y,p)$ such that

$$DF = 0\,, \qquad F|_{y=0} = f$$

giving a covariant $*$ product:

$$f * g := (F \circ G)\Big|_{y=0}$$

Same applies to wave functions: $\phi(x)$ lifts to $\Phi(x,y)$ such that

$$(\nabla - dx^\mu \frac{\partial}{\partial y^\mu} + \rho(\mathbf{r}))\Phi = 0\,, \qquad \Phi\Big|_{y=0} = 0$$

The covariant symbol map is then given by

$$\widehat{f}\phi := (\rho(\mathbf{r})\Phi)\Big|_{y=0}\,, \quad \rho(F) - \text{ Weyl symbol map in } y\text{-space}$$

The inner product:

$$\langle \phi, \chi \rangle = \int d^d x \sqrt{g}\; \phi^*(x)\,\chi(x)$$

Real symbols correspond to formally-hermitean operators.

The second ingredient: a version of the background field method: keep two copies of the spin 2 field: the background $g^{\mu\nu}$ (that also determines quantization) and the perturbation h_2.

The covariant scalar field action:

$$S[g,h,\phi] = \int d^dx \sqrt{g}\ \phi^* \left(\widehat{g} + \mathcal{R}[g]\right) \phi + \int d^dx \sqrt{g}\ \phi^* \widehat{h} \phi \,,$$

It is by construction diffeomorphism-invariant ($\xi = \xi^\mu(x) \frac{\partial}{\partial x^\mu}$)

$$\delta g = \mathcal{L}_\xi g \,, \qquad \delta h = \mathcal{L}_\xi h \,, \qquad \delta\phi = \xi\phi$$

Moreover, in addition to HS gauge transformations

$$\delta h = [g + \mathcal{R} + h, \epsilon]_\star + \{g + \mathcal{R} + h, \omega\}_* \,,$$
$$\delta\phi = (-\widehat{\epsilon} - \widehat{\omega})\phi \,,$$

there are “deformed” Weyl transformations

$$\delta_{\omega_0} g^{\mu\nu} = 2\omega_0 g^{\mu\nu} \,, \qquad \delta_{\omega_0} h = 2\omega_0 h + \delta'_{\omega_0} h \,, \qquad \delta_{\omega_0}\phi = (\frac{d}{2} - 1)\omega_0 \phi$$

$\delta'_{\omega_0} h_s$ is proportional to h_t with $t > s$.

The log-divergent part $S_{CHS}[g,h]$ of the effective action

$$\int D\phi e^{S[\phi,g,h]}$$

should be invariant under the diffeomorphims, deformed Weyl, and HS gauge transformations.

Expand in powers of h

$$S_{CHS}[g,h] = S_0[g] + S_1[g,h] + S_2[g,h] + \dots$$

$$S_1 = \sum_s \int d^dx \sqrt{g}\, K_{\mu_1\dots\mu_s}[g]\; h^{\mu_1\cdots\mu_s}\,, \qquad S_2 = \sum_{s,s'} \int d^dx\; h_s\; O_{ss'}[g]\; h_{s'}$$

It follows $S_0[g]$ is both diffeomorphims and Weyl invariant.

Restrict to $d = 4$ in what follows. Then $S_0[g] \sim \int d^4x \sqrt{g} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ i.e. is a usual conformal gravity action.

Let us concentrate on $S_1 = \sum_s \int K_{\mu_1\dots\mu_s}[g] h^{\mu_1\cdots\mu_s}$. If $K_{\mu_1\dots\mu_s} = 0$ then $g = g_0, h = 0$ is a vacuum solution provided g_0 is a solution to S_0 EOM's. Then S_2 is gauge invariant w.r.t. linearized gauge symmetries.

The gauge symmetries imply that $K_{\mu_1\dots\mu_s}$ are tensors and K_μ is a Weyl-invariant of weight -2. Such tensors are known as admissible invariants.

In particular (d=4):

$K_\mu = 0$

$K_{\mu\nu} \sim B_{\mu\nu} \equiv \nabla^\rho \nabla_\mu P_{\nu\rho} - \nabla^\rho \nabla_\rho P_{\mu\nu} + P^{\rho\sigma} C_{\mu\rho\nu\sigma}$ Bach tensor

$P_{\mu\nu} \equiv \frac{1}{2}\Big(R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu}\Big)$

$K_{\mu\nu\rho} \sim E_{\mu\nu\rho}$ Eastwood–Dighton tensor

In spinor conventions: $E_{ABCA'B'C'} = \Psi_{ABCD}\nabla^{DD'}\Psi_{A'B'C'D'} - \Psi_{A'B'C'D'}\nabla^{DD'}\Psi_{ABCD}\,.$

More generally: Weyl-invariant tensors depending on the metric and its derivatiaves can be represented as polynomials in

$$g_{\mu\nu}, \;\; C_{\mu\nu\rho\sigma}, \;\; \mathcal{D}_\alpha C_{\mu\nu\rho\sigma}, \;\; \mathcal{D}_\alpha \mathcal{D}_\beta C_{\mu\nu\rho\sigma}, \quad \ldots$$

with indices properly contracted by $g^{\mu\nu}$. $\mathcal{D}_\mu$ – Weyl covariant derivative. *Boulanger, 2005*

earlier works by *Gover,...*.

Suppose that $K_{\mu_1\ldots\mu_r} = 0$ for $r < s$. It follows, $K_{\mu_1\ldots\mu_s}$ is a Weyl invariant and satisfies:

$$K^{\nu}{}_{\nu\mu_3\ldots\mu_s} = 0\,, \qquad\qquad \nabla^{\mu_1} K_{\mu_1\mu_2\ldots\mu_s} = 0$$

To first order in curvatures $K_{\mu_1\ldots\mu_s}$ is proportional to (derivatives) of the Bach tensor i.e. vanish on solutions to conformal gravity equations of motion.

This implies that CHS fields consistently propagate on such backgrounds at this order in curvatures, extending the spin 3 result of *Nutma, Taronna 2014*

If in addition Eastwood-Dighton tensor vanishes (this means that the metric is conformal Einstein) the spin 3 is likely consistent to all orders in curvature. However, in its present form our method is not sufficient to prove this.

Mixing of different spins

The essentially new feature over non C-flat background: nontrivial mixing between different spins.

Example: spin 1 and spin 3

linearized gauge transformations with parameters $\epsilon^{\mu\nu}$ act on spin 1:

$$\delta h_\mu = R_{\mu\nu}{}^{\rho}{}_{\sigma} \nabla_\rho \epsilon^{\nu\sigma}$$

In general this mixing can't be removed by a field/gauge-parameter redefinition.

This issue deserves further study.

Conclusions

- Geometrically covariant formulation of nonlinear CHS gauge symmetries. Deformed Weyl symmetry.

- The problem of consistency on non C-flat backgrounds is reformulated as a well-defined mathematical problem of tensor Weyl invariants.

- The known spin 3 consistency at linear order in curvatures is extended to all spins.

- Possible exact results for $s = 3$?