Current Interactions from Nonlinear Higher-Spin Equations

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Higher derivatives in HS interactions

HS interactions contain higher derivatives Bengtsson, Bengtsson, Brink (1983)

Nonanaliticity in Λ via dimensionless combination $\Lambda^{-\frac{1}{2}}\frac{\partial}{\partial x}$ (Fradkin, MV 1987)

By a seemingly local field redefinition it is possible to get rid of currents from HS field equations including the stress tensor (Prokushkin, MV 1998)

$$\phi \to \phi' = \phi + \sum_n a_{nm}(\rho D)^n \phi\,(\rho D)^m \phi + \ldots\,,$$

ρ is the AdS radius, D is the space-time covariant derivative.

The problem: find restrictions on a_{nm} distinguishing between truly non-local and generalized local field redefinitions containing an infinite number of terms but a_{nm} decreasing fast enough with n and m.

The problems in AdS_d and Minkowski space are essentially different

Locality versus Nonlocality

For a massive field equation

$$(\Box + m^2)\phi = 0$$

Greens function can be represented in the pseudolocal form

$$G = (\Box + m^2)^{-1} = m^{-2} \sum_{n=0}^{\infty} \left(-\frac{\Box}{m^2} \right)^n$$

Constant expansion coefficients imply nonlocality.

m^2 is a counterpart of Λ for massless particles in AdS

The problem is to look for a class of field redefinitions which

- are closed under multiple application: form an algebra

- rule out obviously nonlocal field redefinitions like those resulting from Greens functions

Nonlinear HS equations

$$\mathcal{W}(Z;Y;k,\bar{k}|x) = (\mathsf{d} + W) + S\,, \qquad W = dx^n W_n\,, \qquad S = \theta^\alpha S_\alpha + \bar{\theta}^{\dot\alpha}\bar{S}_{\dot\alpha}$$ **1992**

$$\mathcal{W}\star\mathcal{W} = i(\theta^A\theta_A + \eta\theta^\alpha\theta_\alpha B\star k\star\kappa + \bar{\eta}\bar{\theta}^{\dot\alpha}\bar{\theta}_{\dot\alpha}B\star\bar{k}\star\bar{\kappa})$$

$$\mathcal{W}\star B = B\star\mathcal{W}\,, \qquad B = B(Z;Y;k,\bar{k}|x)$$

HS star product

$$(f\star g)(Z;Y) = \frac{1}{(2\pi)^4}\int d^4U\, d^4V \exp\,[iU_A V^A]\, f(Z+U;Y+U)g(Z-V;Y+V)$$

$$\kappa = \exp iz_\alpha y^\alpha\,, \qquad \bar{\kappa} = \exp i\bar{z}_{\dot\alpha}\bar{y}^{\dot\alpha}$$

Massless fields

$$\mathcal{W}(Z;Y;k,\bar{k}|x) = \mathcal{W}(Z;Y;-k,-\bar{k}|x),\; B(Z;Y;k,\bar{k}|x) = -B(Z;Y;-k,-\bar{k}|x)$$

Topological fields

$$\mathcal{W}(Z;Y;k,\bar{k}|x) = -\mathcal{W}(Z;Y;-k,-\bar{k}|x),\; B(Z;Y;k,\bar{k}|x) = B(Z;Y;-k,-\bar{k}|x)$$

Perturbative analysis

The standard vacuum solution is $B=0$ and

$$\mathcal{W}_0 = \mathsf{d}_x + Q + W_0(Y|x)\,, \qquad Q := \theta^A Z_A$$

The space-time one-form $W_0(Y|x)$ solves the flatness equation

$$AdS: \qquad \mathsf{d}_x W_0(Y|x) + W_0(Y|x) \star W_0(Y|x) = 0\,.$$

The star-commutator with Q yields de Rham derivative in Z^A

$$Q \star f(Z;Y) - (-1)^{deg_f} f(Z;Y) \star Q = -2i\mathsf{d}_Z f(Z;Y)\,, \qquad \mathsf{d}_Z = \theta^A \frac{\partial}{\partial Z^A}$$

Standard homotopy formula:

$$\mathsf{d}_Z g(\theta_Z;Z;Y) = f(\theta_Z;Z;Y) \quad \Longrightarrow \quad g(\theta_Z;Z;Y) = \partial^*_Z f + \mathsf{d}_Z \varepsilon + g(0;0;Y)$$

$\mathsf{d}_Z\varepsilon$: **exact forms**

$g(0;0;Y)$: **de Rham cohomology**

Dynamical fields in de Rham cohomology:

$$C(Y;k,\bar{k}|x) = B(0;Y;k,\bar{k}|x)\,, \qquad \omega(Y;k,\bar{k}|x) = W(0;Y;k,\bar{k}|x)$$

Fields and Currents

Spin s is described by the 1-forms $\omega(y,\bar{y}|x)$ and 0-form $C(y,\bar{y}|x)$ obeying

$$\omega(\mu y,\mu\bar{y}\mid x)=\mu^{2(s-1)}\omega(y,\bar{y}\mid x)\,,\qquad C(\mu y,\mu^{-1}\bar{y}\mid x)=\mu^{\pm 2s}C(y,\bar{y}\mid x)$$

Generalized Weyl tensors $C(y,0|x)$ and $C(0,\bar{y}|x)$ describe gauge invariant combinations of derivatives of the gauge fields of spins $s\geq 1$ and matter fields of spins $s=0,1/2$

$C(y,0|x)$ and $C(0,\bar{y}|x)$ are primaries of the Weyl module formed by $C(y,\bar{y}|x)$

Higher powers in y and $\bar{y}$ for a given spin contain higher derivatives

Conserved currents $J(Y_1,Y_2|x)$ are associated with the bilinears of $C(Y|x)$

$$J(Y_1,Y_2|x):=C(Y_1|x)\tilde{C}(Y_2|x)\,,\qquad \tilde{C}(y,\bar{y}|x)=C(-y,\bar{y}|x)\,.$$

As a consequence of the rank-one equation for $C(Y|x)$, the current $J(Y_1,Y_2|x)$ obeys the rank-two equation

Gelfond, MV (2003)

$$\tilde{D}_2J(Y_1,Y_2|x)=0\,,\qquad \tilde{D}_2:=D^L-i\lambda h^{\alpha\dot{\beta}}\Big(y_{1\alpha}\bar{y}_{1\dot{\beta}}-y_{2\alpha}\bar{y}_{2\dot{\beta}}-\frac{\partial^2}{\partial y_1^\alpha\partial\bar{y}_1{}^{\dot{\beta}}}+\frac{\partial^2}{\partial y_2^\alpha\partial\bar{y}_2{}^{\dot{\beta}}}\Big)$$

Current deformation

Current deformation can be formulated as a linear system

$$D\omega + L(w,C) + \Gamma_{cur}(w,J) = 0\,,$$

$$\tilde{D}C + \mathcal{H}_{cur}(w,J) = 0\,, \qquad \tilde{D}_2 J(Y_1,Y_2|x) = 0$$

$$L(w,C) := i\Big(\eta \overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\ \bar{C}(0,\overline{y}|x) + \bar\eta H^{\alpha\beta}\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\ C(y,0|x)\Big)$$

Linear functionals Γ and $\mathcal{H}$ should obey the compatibility conditions

The freedom in $\Gamma_{cur}(w,J)$ and $\mathcal{H}_{cur}(w,J)$ results from field redefinitions

$$\omega \to \omega' = \omega + \Omega(w,J)\,, \qquad C \to C' = C + \Phi(J)\,.$$

Nontrivial $\Gamma_{cur}(w,J)$ and $\mathcal{H}_{cur}(w,J)$ cannot be removed by a field redefinition. Usual current interactions are nontrivial. Schematically,

$$J = J_0 + \Delta J\,,$$

where ΔJ is an improvement that can be removed by a field redefinition.

Concept of (non)triviality of the currents depends on the class of field redefinitions

Unfolded form of usual current interactions

For simplicity: 0-form sector Gelfond, MV (2010)

$$\mathcal{H}_{cur}(w,J)=\frac{1}{4}\int_0^1 d\tau\sum_{h_1,h_2,h_J}a(h_1,h_2,h_J)\int\frac{d\bar{s}d\bar{t}}{(2\pi)^2}\exp i[\bar{s}_{\dot\beta}\bar{t}^{\dot\beta}]$$

$$h(y,\tau\bar{s}+(1-\tau)\bar{t})J_{h_1,h_2,h_J}(\tau y,-(1-\tau)y,\bar{y}+\bar{s},\bar{y}+\bar{t})+c.c.\,,$$

$$h(u,\bar{u})=h^{\alpha\dot\alpha}u_\alpha\bar{u}_{\dot\alpha}$$

J_{h_1,h_2,h_J} **is the projection of** J **to the helicities** h_1,h_2,h_J.

Coefficients $a(h_1,h_2,h_J)$ **remain undetermined at this level.**

$\mathcal{H}_{cur}(w,J)$ **is local, containing a finite number of terms for any** $h_1,h_2,\ h_J$.

$\mathcal{H}_{cur}(w,J)$ **properly reproduces usual current interactions** Gelfond, MV 2010

Locality in the twistor variables

Technically, locality is due to the absence of integration over s and t.

$$\int \frac{dsdt}{(2\pi)^2} \exp i[s_\beta t^\beta] f(y+s,\bar{y}) g(y+t,\bar{y}) = f(y,\bar{y}) \exp[-i \overleftarrow{\partial}_\alpha \overrightarrow{\partial}_\beta \epsilon^{\alpha\beta}] g(y,\bar{y})$$

$$\int \frac{d\bar{s}d\bar{t}}{(2\pi)^2} \exp i[\bar{s}_{\dot\beta} \bar{t}^{\dot\beta}] f(y,\bar{y}+\bar{s}) g(y,\bar{y}+\bar{t}) = f(y,\bar{y}) \exp[-i \overleftarrow{\partial}_{\dot\alpha} \overrightarrow{\partial}_{\dot\beta} \epsilon^{\dot\alpha\dot\beta}] g(y,\bar{y})$$

For given helicities carried by g and f, only a single term in the sum contributes hence containing a finite number of derivatives.

When both integrations are present, the number of derivatives in y and $\bar{y}$ can be infinitely increased without affecting the helicities carried by g and f, implying appearance of infinite tails of derivatives and hence nonlocality.

Current deformation from nonlinear equations

In the 0-form sector the deformation is

$$D_0 C + [\omega\,, C]_* + \mathcal{H}(w, J) = 0\,,$$

$$J(y_1, y_2; \bar{y}_1, \bar{y}_2; K|x) = C(y_1, \bar{y}_1; k, \bar{k}|x) C(y_2, \bar{y}_2; k, \bar{k}|x)$$

A simple computation using the new technique Didenko, Misuna, MV 2015

$$\mathcal{H}(w, J) = \mathcal{H}_\eta(w, J) + \mathcal{H}_{\bar{\eta}}(w, J)\,,$$

$$\begin{aligned}\mathcal{H}_\eta(w, J) = & -\frac{i}{2}\eta \int \frac{dSdT}{(2\pi)^4} \exp iS_A T^A \int_0^1 d\tau \\ & [h(s, \tau\bar{y} - (1-\tau)\bar{t}) J(\tau s, -(1-\tau)y + t; \bar{y} + \bar{s}, \bar{y} + \bar{t}; k, \bar{k}) \\ & - h(t, \tau\bar{y} - (1-\tau)\bar{s}) J((1-\tau)y + s, \tau t, \bar{y} + \bar{s}; \bar{y} + \bar{t}; k, \bar{k})] * k\end{aligned}$$

This deformation is not local, containing integrations over s, t and $\bar{s}, \bar{t}$.

Here h depends on $\bar{y}$ instead of y in the local current deformation.

The two terms result from those in the commutator $[\mathcal{W}, B]_*$

Field redefinition

To reproduce standard current interactions we have to find a field redefinition

$$C \to C'(Y;k,\bar{k}|x) = C(Y;k,\bar{k}|x) + \Phi(Y;k,\bar{k}|x)$$

with Φ linear in J bringing $\mathcal{H}(w,J)$ to $\mathcal{H}_{cur}(w,J)$

First field redefinition

$$\Phi_{1\eta}(Y;k,\bar{k}|x) = \quad \eta \int \frac{dSdT}{(2\pi)^4} \exp iS_A T^A \int d\tau_i \prod_{i=1}^{3} \theta(\tau_i)\delta\Big(1 - \sum_{i=1}^{3} \tau_i\Big)$$
$$\frac{\partial}{\partial \tau_3} J(\tau_3 s + \tau_1 y, t - \tau_2 y; \bar{y} + \bar{s}, \bar{y} + \bar{t}; k, \bar{k}) * k\,,$$

gives

$$D_0 \Phi_{1\eta}(Y;k,\bar{k}|x) = -\frac{i}{2}\eta \int \frac{dSdT}{(2\pi)^4} \exp i[S_A T^A] \int_0^1 d\tau$$
$$\Big[h(s, \tau\bar{y} - (1-\tau)\bar{t}) J(\tau s, -(1-\tau)y + t, \bar{y} + \bar{s}, \bar{y} + \bar{t})$$
$$-h(t, \tau\bar{y} - (1-\tau)\bar{s}) J(s + (1-\tau)y, \tau t, \bar{y} + \bar{s}, \bar{y} + \bar{t})$$
$$-ih(\partial_1 - \partial_2, (1-\tau)\bar{t} + \tau\bar{s}) J(\tau y, -(1-\tau)y; \bar{y} + \bar{s}, \bar{y} + \bar{t}; k, \bar{k})\Big] * k$$

Reincarnation of y

The first two terms just give $\mathcal{H}_\eta(w,J)$. As a result,

$$\mathcal{H}_\eta(w,J) = D_0\Phi_{1\eta}(J) + \mathcal{H}'_\eta(w,J)\,,$$

$$\mathcal{H}'_\eta(w,J) = \frac{\eta}{2}\int \frac{d\bar{s}d\bar{t}}{(2\pi)^2}\exp i[\bar{s}_{\dot\alpha}\bar{t}^{\dot\alpha}]\int_0^1 d\tau h(\partial_1-\partial_2,(1-\tau)\bar{t}+\tau\bar{s})$$
$$J(\tau y,-(1-\tau)y;\bar{y}+\bar{s},\bar{y}+\bar{t};k,\bar{k})*k\,.$$

Being free of integration over s and t, $\mathcal{H}'_\eta(w,J)$ is local but contains one extra space-time derivative compared to $\mathcal{H}_{cur}$. A local field redefinition

$$\Phi_{2\eta} = \frac{i}{2}\eta\int\frac{d\bar{s}d\bar{t}}{(2\pi)^2}\exp i[\bar{s}_{\dot\beta}\bar{t}^{\dot\beta}]\int_0^1 d\tau J(\tau y,-(1-\tau)y;\bar{y}+\bar{s},\bar{y}+\bar{t};k,\bar{k})*k$$

then yields the current deformation with

$$a(h_1,h_2,h_J)=\eta\,,\qquad \bar{a}(h_1,h_2,h_J)=\bar\eta$$

$$\mathcal{H}_\eta(w,J) = \mathcal{H}_{\eta\,cur}(w,J) + D_0(\Phi_{1\eta}+\Phi_{2\eta})$$

$$\mathcal{H}_{\eta\,cur}(w,J) = \frac{\eta}{4}\int\frac{d\bar{s}d\bar{t}}{(2\pi)^2}\exp i[\bar{s}_{\dot\beta}\bar{t}^{\dot\beta}]\int_0^1 d\tau h(y,\tau\bar{s}+(1-\tau)\bar{t})$$
$$J(\tau y,(\tau-1)y;\bar{y}+\bar{s},\bar{y}+\bar{t};k,\bar{k})*k\,.$$

The phase (in)dependence

An important consequence of the flip of chirality $\bar{y} \to y$ is that the current contributions are proportional to $\eta\bar{\eta}$.

Spin-one example

$$R_1(y,\overline{y}|x) = i\Big(\eta\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\overline{y}^{\dot\alpha}\partial\overline{y}^{\dot\beta}}\ \bar{C}(0,\overline{y}|x) + \bar\eta H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\ C(y,0|x)\Big)$$

shift of the first and second terms are proportional to $\bar\eta$ **and** η, **respectively**.

Contribution to *r.h.s.* **of the Maxwell equations is proportional to** $\eta\bar\eta$.

Comparison with previous attempts

For functions

$$f(y,\bar{y}|x) = \frac{1}{2i}\sum_{n,m=0}^{\infty}\frac{1}{n!m!}y_{\alpha_1}\dots y_{\alpha_n}\bar{y}_{\dot{\beta}_1}\dots\bar{y}_{\dot{\beta}_m}f^{\alpha_1\dots\alpha_n}{}_{,}{}^{\dot{\beta}_1\dots\dot{\beta}_m}(x)$$

Star product is

$$(f*g)f_{\alpha(n)}{}_{,\dot{\alpha}(m)} \sim \sum_{p,q,k,l,s,r=0}^{\infty}\delta^n_{p+q}\delta^m_{k+l}\frac{1}{p!q!k!l!s!t!}f_{\alpha(p)\gamma(s)}{}_{,\dot{\alpha}(l)\dot{\gamma}(t)}g_{\alpha(q)}{}^{\gamma(s)}{}_{,\dot{\alpha}(k)}{}^{\dot{\gamma}(t)}\,.$$

For the field redefinition induced by Φ_1

$$\Phi_{1\eta}(Y;k,\bar{k}|x) = \int\delta(1-\sum_{j=1}^{3}\tau_i)\prod_{i=1}^{3}d\tau_i\theta(\tau_i)\delta(1-\sum_{j=1}^{3}\tau_i)$$
$$\frac{\partial}{\partial\tau_3}J(\tau_3 s+\tau_1 y, t-\tau_2 y;\bar{y}+\bar{s},\bar{y}+\bar{t};k,\bar{k})*k$$

Integration over homotopy parameters τ_i over a triangle $\sum_{j=1}^{3}\tau_i = 1$

softens the coefficients in $\Phi_{1\eta}(Y;K|x)$ compared to the star product

$$\frac{1}{p!q!k} \to \frac{p}{(p+k+l+2)!}\int_0^1 dt_1 t_1^{a_1}\dots\int_0^1 dt_p t_p^{a_p}\delta(\sum_i t_i - 1) = \frac{\prod_{i=1}^{p} a_i!}{(\sum_{i=1}^{p} a_i + p - 1)!}$$

implies $\varepsilon = 1$.

HS holography

The phase φ of η should be related to the Chern-Simons coupling of the boundary vector model. Does this fit the conclusion that the HS cubic vertex is φ-independent?

$$C^{j\,1-j}(y,\bar{y}|\mathbf{x},\mathbf{z}) = \mathbf{z}\exp(y_\alpha\bar{y}^\alpha)T^{j\,1-j}(w,\bar{w}|\mathbf{x},\mathbf{z})\,, \qquad w^\alpha = \mathbf{z}^{1/2}y^\alpha \quad \bar{w}^\alpha = \mathbf{z}^{1/2}\bar{y}^\alpha$$

where $T^{j\,1-j}$ are associated with the boundary currents.

The contribution of HS connections at the boundary cannot be neglected except for the boundary conditions MV 2012

$$\bar{\eta}T_{+}^{j\,1-j}(y,\bar{y}|\mathbf{x},0) - \eta T_{-}^{1-j\,j}(i\bar{y},iy|\mathbf{x},0) = 0\,,$$

where T_+ and T_- are the positive and negative helicity parts of $T(y,\bar{y}|x)$.

In terms of remaining real boundary fields

$$j^j(y,\bar{y}|\mathbf{x}) := \frac{1}{2}\Big(\bar{\eta}T_{+}^{j\,1-j}(y,\bar{y}|\mathbf{x},0) + \eta T_{-}^{1-j\,j}(i\bar{y},iy|\mathbf{x},0)\Big) = \bar{\eta}T_{+}^{j\,1-j}(y,\bar{y}|\mathbf{x},0)$$

the final result matches the form of the deformation of the HS current algebra found by Maldacena and Zhiboedov

$$V = cos^2(\varphi)V_b + sin^2(\varphi)V_f + \frac{1}{2}sin(2\varphi)V_o$$

Conclusion

Nonlinear HS equations properly reproduce the HS current interactions with the φ-independent coupling constant.

Explicit form of the appropriate field redefinition suggests a proper form of generalized local field redefinitions

Proper dependence on the phase parameter in the holographic duals of the AdS_4 HS theory is reproduced by the phase-independent vertex of the bulk theory HS theory via phase-dependent boundary conditions.

Green light for the analysis of HS field equations

Invariant functionals

String-like HS theory

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Phase dependence via boundary conditions

The contribution of HS connections at the boundary cannot be neglected except for the boundary conditions MV 2012

$$\bar{\eta} T_{+}^{j\,1-j}(y,\bar{y}|\mathbf{x},0) - \eta T_{-}^{1-j\,j}(i\bar{y},iy|\mathbf{x},0) = 0\,,$$

where T_+ and T_- are the positive and negative helicity parts of $T(y,\bar{y}|x)$.

Upon imposing boundary conditions, remaining real boundary fields are

$$j^j(y,\bar{y}|\mathbf{x}) := \frac{1}{2}\Big(\bar{\eta} T_{+}^{j\,1-j}(y,\bar{y}|\mathbf{x},0) + \eta T_{-}^{1-j\,j}(i\bar{y},iy|\mathbf{x},0)\Big) = \bar{\eta} T_{+}^{j\,1-j}(y,\bar{y}|\mathbf{x},0)\,.$$

Independence of the bulk HS vertex on φ implies that the boundary vertex has the structure

$$V = \sum_{i,j=1,2} (a_{ij} T_{+}^{i\,1-i} T_{+}^{j\,1-j} + b_{ij} T_{-}^{i\,1-i} T_{-}^{j\,1-j} + e_{ij} T_{-}^{i\,1-i} T_{+}^{j\,1-j})\,,$$

where a_{ij}, b_{ij} and e_{ij} are some φ-independent coefficients built from components of the boundary HS connections and background fields.

In terms of real φ-independent currents V reads

$$V = \frac{1}{\eta\bar{\eta}} \sum_{i,j=1,2} (\exp 2i\varphi\, a_{ij} j_+^{i\,1-i} j_+^{j\,1-j} + \exp -2i\varphi\, b_{ij} j_-^{i\,1-i} j_-^{j\,1-j} + e_{ij} j_-^{i\,1-i} j_+^{j\,1-j})\,.$$

Manifest dependence on φ identifies the parity even boson $(\varphi = 0)$ **vertex** V_+ **and fermion** $(\varphi = \pi/2)$ **vertex** V_-

$$V_\pm = \frac{1}{\eta\bar{\eta}} \sum_{i,j=1,2} (\pm a_{ij} j_+^{i\,1-i} j_+^{j\,1-j} \pm\, b_{ij} j_-^{i\,1-i} j_-^{j\,1-j} + e_{ij} j_-^{i\,1-i} j_+^{j\,1-j})\,,$$

Since parity transformation exchanges the positive and negative helicities, the remaining parity-odd vertex is

$$V_o = \frac{i}{\eta\bar{\eta}} \sum_{i,j=1,2} (a_{ij} j_+^{i\,1-i} j_+^{j\,1-j} -\, b_{ij} j_-^{i\,1-i} j_-^{j\,1-j})\,.$$

This gives the following formula matching the form of the deformation of the HS current algebra found by Maldacena and Zhiboedov

$$V = cos^2(\varphi) V_b + sin^2(\varphi) V_f + \frac{1}{2} sin(2\varphi) V_o\,,$$