Presymplectic structures and intrinsic Lagrangians

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Based on:

M.G., to appear

M.G., A. Verbovetsky, to appear

K. Alkalaev, M.G. 2013

Also:

M.G. 2012

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HSTH-4, LPI, Moscow

Motivations

- Lagrangians (or their substitutes) are inevitable for quantization

- Existence of a Lagranian formulation is often considered as a selection criterium

- Analysis becomes problematic once auxiliary fields are in the game. Ex.: unfolded formulation of HS theores etc.

- Lack of an invariant understanding of the structures underlying Lagrangian formulation

Jet space

Space-time coordinates (independent variables): x^a, $a = 1, \ldots, n$.

Fields (dependent variable) ϕ^i.

$$J^0 : \quad x^a, \phi^i, \qquad J^1 : \quad x^a, \phi^i, \phi^i_a\,, \qquad J^2 : \quad x^a, \phi^i, \phi^i_a, \phi^i_{ab}\,, \qquad \ldots$$

Projections:

$$\ldots \to J^N \to J^{N-1} \to J^{N-2} \to \ldots \to J^1 \to J^0$$

Useful to work with J^∞. A local diff. form on J^∞ – a form on J^N for some N seen as that on J^∞.

J^∞ is equipped with the total derivative

$$\partial^T_a = \frac{\partial}{\partial x^a} + \phi^i_a \frac{\partial}{\partial \phi^i} + \phi^i_{ab} \frac{\partial}{\partial \phi^i_b} + \ldots$$

For a given field configuration $\phi^i = s^i(x)$ and local function $f[\phi]$

$$(\partial^T_a f)\Big|_{\phi=s, \phi_a=\partial_a s, \ldots} = \partial_a (f\Big|_{\phi=s, \phi_a=\partial_a s, \phi_{ab}=\partial_a\partial_b s, \ldots})$$

Space time differentials dx^a. Horizontal differential

$$Q \equiv d_H = dx^a \partial_a^T \,, \qquad Q^2 = 0\,.$$

Differential forms:

$$\alpha = \alpha(x, dx, \phi, \phi_a, \ldots)_{I_1 \ldots I_k} d_{\mathsf{V}} \phi^{I_1} \ldots d_{\mathsf{V}} \phi^{I_k} \,, \qquad \phi^I = \phi^i_{a_1 \ldots a_m}$$

Vertical De Rham differential:

$$d_{\mathsf{V}} = d - Q = d_{\mathsf{V}} \phi^I \frac{\partial}{\partial \phi^I}$$

Variational bicomplex:

$$d_{\mathsf{V}}^2 = 0, \qquad d_{\mathsf{V}} Q + Q d_{\mathsf{V}} = 0\,, \qquad Q^2 = 0$$

Bidegree (l, p).

A system of partially differential equations (PDE) is a collection of local functions

$$E_\alpha[\phi, x] \,.$$

The equation manifold (stationary surface) is $\mathcal{E} \subset J^\infty$ singled out by:

$$\partial^T_{a_1} \dots \partial^T_{a_l} E_\alpha = 0 \,, \qquad l = 0, 1, 2, \dots$$

understood as the algebraic equations in J^∞. It is usually assumed that x^a, ϕ^i are not constrained, e.g. $\mathcal{E}$ is a bundle over the space-time.

∂^T_a are tangent to $\mathcal{E}$ and hence restricts to $\mathcal{E}$. So do the differentials Q and d_{V}. $\partial^T_a|_\mathcal{E}$ determine a dim-n integrable distribution (Cartan distribution).

Definition: *[Vinogradov] A PDE is a manifold $\mathcal{E}$ equipped with an integrable distribution.*

In addition one typically assumes regularity, constant rank, and that $\mathcal{E}$ is a bundle over the spacetime. Use notation $(\mathcal{E}, Q)$.

In this form it is clear which PDEs are to be considered isomorphic.

Scalar field Example: Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \ldots$. Already ϕ_{ab} are not independent. One can e.g. take $\phi_{abc\ldots}$ traceless. The Q differential on $\mathcal{E}$ reads as

$$Qx^a = dx^a\,, \qquad Q\phi = dx^a\phi_a,\,, \qquad Q\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial \phi})\,, \qquad \ldots$$

So if the system is nonlinear, i.e. $\frac{\partial V}{\partial \phi}$ nonlinear in ϕ, Q is also nonlinear.

Intrinsic (unfolded) realization

Given PDE $(\mathcal{E}, Q)$ defined invariantly one can always find a jet space $\mathcal{J}$ such that $(\mathcal{E}, Q)$ can be realized as a stationary surface of some $E_\alpha[u, x]$.

There is an intrinsic way to realize $(\mathcal{E}, Q)$ explicitly. If x^a, ψ^A coordinates on $\mathcal{E}$ (e.g. $\psi^A = \{\phi, \phi_a, \phi_{ab}, \ldots\}$) promote ψ^A to fields $\psi^A(x)$ of a new theory and subject them to EOM's

$$\boldsymbol{d}\psi^A = Q\psi^A \,, \qquad \text{components: } \frac{\partial}{\partial x^a}\psi^A(x) = (\partial_a^T \psi^A)(x)$$

Proposition: *The original PDE $(\mathcal{E}, Q)$ is equivalent to $\boldsymbol{d}\psi^A = Q\psi^A$*

Comments:

- Version of the unfolded formulation (though only zero forms). Unfolded form of gauge systems involves gauge form-fields. *Vasiliev, 1987,. . .*

- Generalized version of the Proposition involving gauge forms and BRST extension was formulated and proved using BRST technique and Koszule-Tate differential. *Barnich, M.G.,Semikhatov, Tipunin 2004, Barnich, M.G 2010*

Jet space formulation

Becuase $\mathcal{E}$ is a bundle over spacetime, take $\mathcal{J}^{new} = J^{\infty}(\mathcal{E})$. More precisely, if x^a, dx^a, ψ^A are coordinates on $\mathcal{E}$ then

$$x^a, \quad dx^a, \quad \psi^A, \quad \psi^A_b, \quad \psi^A_{bc}, \quad \psi^A_{bcd}, \quad \ldots$$

are coordinates on $\mathcal{J}^{new}$.

New jet space is equipped with its own horizontal differential:

$$D_H = dx^a(\frac{\partial}{\partial x^a} + \psi^A_a \frac{\partial}{\partial \psi^A} + \psi^A_{ab} \frac{\partial}{\partial \psi^A_b} + \ldots)$$

"Old" differential Q on $\mathcal{E}$ is extended to $\mathcal{J}^{new}$ by $[D_H, Q] = 0$.

In the new jet space $\mathcal{J}^{new}$ consider the following PDE

$$D_H \psi^A = Q \psi^A$$

In this form the new PDE is manifestly isomorphic to $(\mathcal{E}, Q)$ (because manifolds are isomorphic and horizontal differentials are equal by construction)

Variational (Lagrangian) equations

Let us get back to equations $E_i[\phi, x] = 0$ on the jet space J^∞. These are said variational (Lagrangian) if

$$\mathcal{E}_i = \frac{\delta^{EL} L}{\delta \phi^i}\,, \qquad \frac{\delta^{EL} F[u,x]}{\delta \phi^i} \equiv \frac{\partial F}{\partial \phi^i} - \partial^T_a \frac{\partial F}{\partial \phi^i_a} + \partial^T_a \partial^T_b \frac{\partial F}{\partial \phi^i_{ab}} - \ldots$$

for some local function $L = L[\phi, x]$. It is convenient to work in terms of Lagrangian density $\mathcal{L} = (dx)^n\, L$.

Here and below

$$(dx)^n = dx^1 \ldots dx^n\,, \qquad (dx)^{n-1}_a = \frac{1}{(n-1)!} \epsilon_{ab_2 \ldots b_n} dx^{b_1} \ldots dx^{b_n}$$

The notion of Lagrangian is explicitly based on the realization of the equation $(\mathcal{E}, Q)$ in terms of the jet space $\mathcal{J}$. For instance it's possible that $\mathcal{E} \subset \mathcal{J}$ is variational while $\mathcal{E} \subset \mathcal{J}'$ is not. Naive invariant object – the restriction of $\mathcal{L}$ to $\mathcal{E}$, does not make much sense.

Presymplectic structure

It is well-known that $\mathcal{L}=(dx)^n L[x,\phi]$ induce an invariant object on $\mathcal{E}$

Crnkovic, Witten, 1987, Hydon 2005,...

$$(dx)^n\, E_i\, d\phi^i = d_{\mathsf{V}}\mathcal{L} - Q\widehat{\chi}\,, \qquad \text{components:} \quad \frac{\delta^{EL}L}{\delta\phi^i} = \frac{\partial L}{\partial\phi^i} + \partial^T_a(\widehat{\chi}^a_i)$$

for some 1 form $\widehat{\chi}=\widehat{\chi}_i d_{\mathsf{V}}\phi^i + \widehat{\chi}_{ia} d_{\mathsf{V}}\phi^i_a + \ldots$ of degree $n-1$, called presymplectic potential. For $\chi=\widehat{\chi}|_{\mathcal{E}}$ we have

$$Q\sigma = 0\,, \qquad \sigma = d\chi$$

So we have conserved closed 2-form on $\mathcal{E}$. It's called canonical presymplectic structure.

As an example consider $L(\phi,\phi_a,\phi_{ab})$. One finds:

$$\chi = (dx)^{n-1}_a \left((\frac{\partial L}{\partial\phi^a} - \partial^T_b \frac{\partial L}{\partial\phi_{ab}}) d_{\mathsf{V}}\phi + \frac{\partial L}{\partial\phi_{ab}} d_{\mathsf{V}}\phi_b \right)\Big|_{\mathcal{E}}$$

In particular, for a scalar field with $L=\frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)$

$$\chi = (dx)^{n-1}_a\ \phi^a\, d_{\mathsf{V}}\phi\,, \qquad \sigma = (dx)^{n-1}_a\ d_{\mathsf{V}}\phi^a\, d_{\mathsf{V}}\phi$$

More generally:

Definition: *A 2-form* σ *of degree* $n-1$ *on* $(\mathcal{E}, Q)$ *is called compatible presymplectic structure if* $Q\sigma = 0\,, d\sigma = 0$*.*

Such form in general can be considered irrespective of any realization in terms of jet space and/or Lagrangian.

Symmetries and conservation laws

A well-known fact: both symmetries and conservation laws can be defined in terms of the equation manifold $(\mathcal{E}, Q)$.

Recall: a vector field $\widehat{V}$ on $\mathcal{J}$ is a symmetry if it is evolutionary i.e. $[Q, \widehat{V}] = 0$ and tangent to $\mathcal{E} \subset \mathcal{J}$.

Intrinsic terms: a vector field V on $(\mathcal{E}, Q)$ satisfying is called symmetry if $[Q, V] = 0$ (typically one also requires $Vx^a = 0$).

If $\mathcal{E} \subset \mathcal{J}$ is variational then variational symmetries restricted to $\mathcal{E}$ satisfy in addition

$$L_V \sigma = 0$$

Conservation law (conserved curent) is a degree $n-1$ 0-form K on $\mathcal{E}$ such that $QK = 0$. K of the form $K = QM$ is trivial.

Any compatible presymplectic structure determines a map from symmetries to conserved currents according to

$$dK = i_V \sigma \,, \qquad \text{components: } \frac{\partial}{\partial \psi^A} K = \sigma_{AB} V^B$$

Note: $d i_V \sigma = L_V \sigma = 0$. Trivial symmetries are mapped to trivial conserved currents. In the Lagrangian case this is Noether theorem. General case was also discussed recently *Sharapov 2016*.

Note that it is different from the Poisson (BV antibracket) bracket map from conservation laws to symmetries. The degenerate version of the bracket is known as Lagrange structure *Lyakhovich, Sharapov*

Suppose that $(\mathcal{E}, Q, \sigma)$ is realized as $\mathcal{E} \subset J^{\infty}$. Then σ determines a Lagrangian form $\mathcal{L}$ on J^{∞} such that EL equations derived from $\mathcal{L}$ are in general consequences of those defining $\mathcal{E}$.

More precisely, if $\mathcal{E}'$ is an equation manifold defined by $\mathcal{L}$ then $\mathcal{E} \subset \mathcal{E}'$. Even if σ is canonical (derived from a Lagrangian) there is no guarantee that constructed $\mathcal{L}$ is equivalent to the starting point Lagrangian.

Khavkine 2012, based on earlier: Bridges, Hydon, Lawson 2009, Hydon 2005

Intrinsic Lagrangian

Given an equation manifold $(\mathcal{E}, Q, \sigma)$ equipped with the compatible presymplectic structure one can construct a natural Lagrangian in terms of the $\mathcal{E}$-valued fields.

First: define generalized Hamiltonian (better BRST charge) which is a conserved current associated to Q seen as a symmetry of $\mathcal{E}$. Degree n function $\mathcal{H}$ on $\mathcal{E}$ defined by

$$d_{\mathsf{V}}\mathcal{H} = i_Q\sigma\,, \qquad \text{components:} \quad \frac{\partial}{\partial\psi^A}\mathcal{H} = \sigma_{AB}Q^B$$

In the Lagrangian case

$$\mathcal{H} = \chi_A Q^A - \mathcal{L}|_{\mathcal{E}} \qquad Q^A = Q\psi^A$$

E.g. in the simple case where $\mathcal{L} = (dx)^n L(\phi, \phi_a)$

$$\chi = (dx)^{n-1}_a \left(\frac{\partial L}{\partial \phi_a} d_{\mathsf{V}}\phi\right)\Big|_{\mathcal{E}}\,, \qquad \mathcal{H} = (dx)^n \left(\frac{\partial L}{\partial \phi_a}\phi_a - L\right)\Big|_{\mathcal{E}}$$

New (intrinsic) Lagrangian:

$$\mathcal{L}^C = i_{\boldsymbol{d}}\chi - \mathcal{H}\,, \qquad \text{components:} \quad \mathcal{L}^C = \chi_A \boldsymbol{d}\psi^A - \mathcal{H}$$

The respective action can be seen as presymplectic generalization

Alkalaev, M.G. 2013

$$S^C = \int \Big(\chi_A(\psi, x, dx)\boldsymbol{d}\psi^A(x) - \mathcal{H}(\psi, x, dx)\Big)$$

of AKSZ action. Its equations of motion read as

$$\sigma_{AB}(\boldsymbol{d}\psi^B - Q^B) = 0\,,$$

and hence are consequences of the original $\boldsymbol{d}\psi^B - Q^B = 0$.

For a local theory $\mathcal{L}^C$ does not depend on most of the fields ψ^A. These can be treated as pure-gauge variables with algebraic (shift) gauge transformations. With this interpretation and under certain assumptions we can prove that starting point $\mathcal{L}$ and $\mathcal{L}^C$ are equivalent.

Examples

Scalar field: Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi) \tag{1}$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \dots$. take $\phi_{abc\dots}$ traceless. The Q differential reads as

$$Qx^a = dx^a\,, \qquad Q\phi = dx^a\phi_a,\,, \qquad Q\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial \phi})$$

The presymplectic potential and 2-form:

$$\chi = \left((dx)^{n-1}_a(\frac{\partial L}{\partial \phi^a} - \partial^T_c\frac{\partial L}{\partial \phi_{ca}})d_{\mathsf{V}}\phi)\right)\Big|_{\mathcal{E}} = (dx)^{n-1}_a\phi^a d_{\mathsf{V}}\phi\,, \quad \sigma = (dx)^{n-1}_a d_{\mathsf{V}}\phi^a d_{\mathsf{V}}\phi$$

The Hamiltonian obtained from $d\mathcal{H} - i_Q\sigma = 0$:

$$\mathcal{H} = (dx)^n(\phi_a\phi^a - L|_{\mathcal{E}}) = \frac{1}{2}\phi^a\phi_a + V(\phi)$$

The intrinsic Larangian: *Schwinger*

$$\mathcal{L}^c = (dx)^n\left(\phi^a(\partial_a\phi - \frac{1}{2}\phi_a) - V(\phi)\right)$$

Polywave equation

The simplest genuine higher derivative example is $L = \frac{1}{2}\Box\phi\Box\phi = \frac{1}{2}\phi_{aa}\phi_{bb}$ (here and below $\phi_{aa} = \eta^{ab}\phi_{ab}$). Presymplectic potential:

$$\chi = (-\phi_{acc}d_{\mathsf{V}}\phi + \phi_{cc}d_{\mathsf{V}}\phi_a)(dx)^{n-1}_a$$

Hamiltonian

$$\mathcal{H} = (dx)^n(-\phi_{acc}\phi_a + \frac{1}{2}\phi_{cc}\phi_{aa})\,.$$

The intrinsic action takes the form

$$S^C = \int d^nx(-\phi_{acc}(\partial_a\phi - \phi_a) + \phi_{cc}\partial_a\phi_a - \frac{1}{2}\phi_{aa}\phi_{cc})\,.$$

Note that the action depends on only the following variables $\phi, \phi_a, \phi_{aa}, \phi_{acc}$ but NOT on the traceless component of ϕ_{ab} and ϕ_{abc}.

It is equivalent to $\int \phi_{aa}\phi_{cc}$. Indeed, varying ϕ_a and ϕ_{acc} gives $\phi_a = \partial_a\phi$ and $\phi_{acc} = \partial_a\phi_{cc}$ resulting in

$$\int d^nx(\phi_{cc}\partial_a\partial_a\phi - \frac{1}{2}\phi_{aa}\phi_{cc})$$

YM theory

The YM field is A^a taking values in a Lie algebra $\mathfrak{g}$ equipped with an invariant inner product $\langle , \rangle$. We will use notation $A^a_{b_1 \dots b_l}$ for $\partial^T_{b_1} \dots \partial^T_{b_l} A^a$. The Lagrangian:

$$L = \frac{1}{4}\langle F_{ab}, F_{ab}\rangle \,, \qquad F_{ab} := A^b_a - A^a_b + [A^a, A^b]\,.$$

Coordinates on $\mathcal{E}$:

$$x^a, A^a, F_{ab}, S_{ab} := A^b_a + A^a_b, A^a_{bc}, \dots$$

The one form χ is given by

$$\chi = \frac{\partial L}{\partial A^b_a} dA^b (dx)^{n-1}_a = \langle F_{ab}, dA^b\rangle (dx)^{n-1}_a$$

The Hamiltonian

$$\mathcal{H} = (\frac{\partial L}{\partial A^b_a} A^b_a - \frac{1}{4}\langle F_{ab}, F_{ab}\rangle)(dx) = \frac{1}{2}\langle F_{ab}, \frac{1}{2} F_{ab} - [A^a, A^b]\rangle$$

The intrinsic action

$$\int \frac{1}{2}\langle F_{ab}, \partial_a A^b - \partial_b A^a\rangle - \frac{1}{2}\langle F_{ab}, \frac{1}{2}F_{ab} - [A^a, A^b]\rangle =$$

$$\int \frac{1}{2}\langle F_{ab}, \partial_a A^b - \partial_b A^a + [A^a, A^b] - \frac{1}{2}F_{ab}\rangle$$

equivalent to the starting point action through the elimination of F_{ab} by its own equations of motion.

Well-known first-order action for YM.

Algebraic gauge symmetries

Assume that starting point Lagrangian $\mathcal{L}$ does not have shift gauge symmetries i.e. there are no invertible nontrivial R_i^α such that

$$R_\alpha^i[\phi, \partial_a^T]\,\frac{\delta^{EL}\mathcal{L}}{\delta\phi^i} = 0\,.$$

The intrinsic Lagrangian does have infinite amount of shift gauge symmetry. EOMs are

$$\sigma_{AB}(\boldsymbol{d}\psi^B - Q^B) = 0$$

so that any null vector of σ_{AB} gives rise to a shift gauge symmetry. If $\sigma_{AB}R^B(\psi) = 0$ then $\delta\psi^A = R^A\epsilon(x)$ is a gauge symmetry of the intrinsic action.

Interpretation of the intrinsic action: all its shift gauge symmetries are taken into account (the respective fields are set to fixed values – i.e. gauge-fixed).

Restrict to “reasonable theories: the Lagrangian theory is “reasonable” if by adding/eliminating auxiliary fields and local invertible change of variables the action $\int \mathcal{L}$ can be brought to the form

$$S^{first} = \int \mathcal{L}^{first}[u] = \int d^dx(V^a_\lambda(u,x)\partial_a u^\lambda - H(u,x))$$

and such that its equations of motion do not imply algebraic constraints between undifferentiated fields u^λ. (i.e. u^λ reduced to $\mathcal{E}$ remain independent)

Note that know frame-like Larangians are “resonable”. Thanks to *Vasiliev, Zinoveiv, Alakalev, Shaynkman, Skvortsov, ...* all known free Lagragian HS fields are resonable.

Proposition: *for a "reasonable" system the original Lagrangian $\mathcal{L}[\phi]$ and the intrinsic Lagrangian $\mathcal{L}^C[\psi]$ are equivalent.*

Proof. Equivalent Lagrangian formulations result in equivalent presymplectic structures on the equation manifold $\mathcal{E}$. It is enough to consider the first order Lagrangian. The respective presymplectic structure reads as

$$\chi = ((dx)_a^{n-1} V_\lambda^a du^\lambda)\Big|_{\mathcal{E}} = (dx)_a^{n-1} V_\lambda^a du^\lambda$$

Hamiltonian:

$$\mathcal{H} = ((dx)^n V_\lambda^a u_a^\lambda - \mathcal{L}^{first})|_{\mathcal{E}} = (dx)^n H$$

Finally:

$$\mathcal{L}^C = (dx)^n (V_\lambda^a \partial_a u^\lambda - H(\phi))$$

and explicitly coincides with the starting point first order Lagrangian. $\square$

BRST extension and frame-like Lagrangians

To make our picture more geometrical let us introduce ghosts:

$$x^a, \psi^A \quad \to \quad x^a, \psi^A, C^\alpha$$

$$Q \equiv d_H \quad \to \quad Q = d_H + \gamma\,, \qquad \gamma = C^\alpha R^A_\alpha(\psi)\frac{\partial}{\partial \psi^A} + C^\alpha C^\beta U^\gamma_{\alpha\beta}(\psi)\frac{\partial}{\partial C^\gamma}$$

$$d_H^2 = 0 = \gamma^2 \qquad d_H\gamma + \gamma d_H = 0$$

Geometrically: $\mathcal{E}$ is now equipped with two integrable distributions: Cartan (d_H) and gauge (γ)

AKSZ sigma model with the target $\mathcal{E}$

$$\psi^A \quad \to \quad \psi^A(x) \qquad\quad C^\alpha \to A^\alpha_a(x) dx^a$$

If $\Psi^I = \{\psi^A, C^\alpha\}$, equations of motion:

$$d\Psi^I = Q^I(\Psi) \quad \to$$

$$dA^\alpha = d_H A^\alpha + U^\gamma_{\alpha\beta}(\psi) A^\gamma A^\beta\,, \qquad\quad d\psi^B = d_H \psi^B + R^B_\alpha(\Psi) A^\alpha$$

(Nonminimal) unfolded formulation *Vasiliev*

Construction and equivalence proof

Barnich, M.G.,Semikhatov, Tipunin 2004, Barnich, M.G 2010

Example of gravity

After elimination of contractible pairs for Q the manifold $\mathcal{E}$

$$e^a, \quad \omega^{ab}, \quad W^{cd}_{ab}, \quad W^{cd}_{ab|e}, \quad W^{cd}_{ab|e\ldots}$$

– ghosts to which frame field and spin connection are associated and Weyl tensor and its covariant derivatives.

$$Qe^a = \omega^a{}_c\, e^c\,, \qquad Q\omega^{ab} = \omega^a{}_c\,\omega^{cb} + e^c e^d W^{ab}_{cd}\,, \qquad \ldots\,,$$

Presymplectic potential χ and form *Alkalaev, M.G. 2013*

$$\chi = \frac{1}{2}\epsilon_{abcd} d\omega^{ab} e^c e^d\,, \qquad \sigma = d\omega^{ab} de^c \epsilon_{abcd} e^d$$

Hamiltonian (term with Weyl tensor vanishes)

$$\mathcal{H} = Q^A \chi_A = \frac{1}{2}\omega^a_c \omega^{cb}\epsilon_{abcd} e^c e^d$$

Intrinsic action (frame-like GR action):

$$S^C = \int \chi_A(\boldsymbol{d}\psi^A + Q^A) = S_{GR}[e,\omega] = \int (\boldsymbol{d}\omega^{ab} + \omega^a{}_c\omega^{cb})\epsilon_{abcd} e^c e^d$$

Conclusions

- A Lagrangian system can be defined in terms of its equation manifold $\mathcal{E}$ without refereeing to any particular realization of $\mathcal{E}$ in one or another set of fields and choice of the Lagrangian. While the structure of the equation is encoded in the differential Q the Lagrangian is encoded in the compatible presymplectic structure σ.

- In particular, when looking for a Lagrangian for an equation $\mathcal{E}$ it is enough to study compatible presymplectic structures on $\mathcal{E}$. No need to study possible explicit realizations of $\mathcal{E}$.

- Easy to see whether Lagrangian systems are equivalent or not.

- BRST extension to manifestly gauge systems. Intrinsic Lagrangian = Frame-like Lagrangian.

- The presymplectic form can be seen to originate from the odd symplectic form of the Batalin-Vilkovisky formalism.

Parent Lagrangian

One way to understand where do the structure of the intrinsic Lagrangian comes from is to consider "parent" action for $L = L(\phi, \phi_a, \phi_{ab})$:

$$S^P = \int \left(L(\phi, \phi_a, \phi_{ab}) + \pi^a(\partial_a \phi - \phi_a) + \pi^{ac}(\partial_a \phi_c - \phi_{ac}) + \ldots \right) .$$

Its equations of motion read as

$$\frac{\partial L}{\partial \phi} - \partial_a \pi^a = 0 ,$$

$$\pi^a - \frac{\partial L}{\partial \phi_a} + \partial_c \pi^{ca} = 0 , \qquad \pi^{ab} - \frac{\partial L}{\partial \phi_{ab}} = 0 , \qquad \pi^{ab\ldots} = 0$$

$$\phi_a = \partial_a \phi , \qquad \phi_{ab} = \partial_{(a} \phi_{b)} , \qquad \ldots$$

Using the last line the derivatives in the first two lines can be replaced with the total derivatives. Using the second line the first equation becomes EL

$$\frac{\partial L}{\partial \phi} - \partial^T_a \frac{\partial L}{\partial \phi_a} + \partial^T_c \partial^T_a \frac{\partial L}{\partial \phi_{ca}} = 0 .$$

Introduce 1-form of degree $n-1$:

$$\bar{\chi} = (dx)_a^{n-1}(\pi^a d\phi + \pi^{ab} d\phi_b + \ldots)$$

"parent" Hamiltonian

$$\bar{\mathcal{H}} = (\pi^a \phi_a + \pi^{ab}\phi_{ab} + \ldots - L(\phi, \phi_a, \phi_{ab}))(dx)^n \,.$$

The parent action can be written as

$$S^P = \int (\bar{\chi}_A \boldsymbol{d}\Psi^A - \bar{\mathcal{H}}) \,,$$

where Ψ^A stand for all the coordinates $\phi, \phi_{\ldots}$, $\pi^{\cdots}$.

Consider the following submanifold of the space of $x^a, dx^a, \phi, \pi^{\cdots}, \phi_{\dots}$

$$\pi^a - \frac{\partial L}{\partial \phi^a} + \partial^T_c \frac{\partial L}{\partial \phi_{ca}} = 0\,, \qquad \pi^{ab} - \frac{\partial L}{\partial \phi_{ab}} = 0\,, \quad \pi^{ab\dots} = 0\,,$$
$$\partial^T_{a_1} \dots \partial^T_{a_k}(EL) = 0\,,$$

These are consequences of the parent action equations of motion.

The submanifold they single out is $\mathcal{E}$ (equation manifold of L).

$$\chi = \bar{\chi}|_{\mathcal{E}} \qquad \text{Presymplectic potential for } L$$

One can show

$$i_Q d\sigma = d\mathcal{H}\,, \qquad \mathcal{H} = \bar{\mathcal{H}}|_{\mathcal{E}}\,, \qquad \sigma = d\chi$$

Furthermore, χ and $\mathcal{H}$ determine the intrinsic action

$$S^C[\psi] = \int \Big(\chi_A(x, dx^a, \psi)\boldsymbol{d}\psi^A - \mathcal{H}(x, dx^a, \psi)\Big)\,,$$

where x^a, ψ^A are coordinates on $\mathcal{E}$. This can be independently arrived at by eliminating auxiliary fields starting from the parent action.