# Conserved spinning fields in conformal field theories

$$\partial_\mu \langle T_{\mu\nu} \ldots \rangle = 0$$

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based on arXiv:1311.4546 and ongoing work

High Spin Theory and Holography, May 30, 2016

# Q: to what extent the full set of Ward identities constraint correlators of conserved currents?

- conformal Ward identities

  invariance under the action of conformal group

$$\sum_i \mathcal{L}^i \langle S_{\mu\ldots}(x_1)\ldots\rangle = 0$$

- diffeomorphism Ward identities

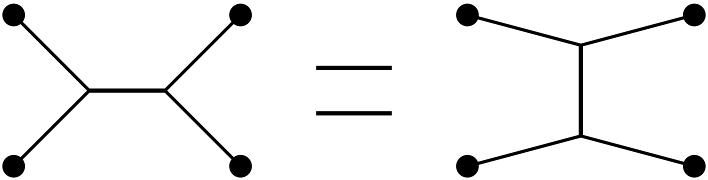
  conservation of stress-energy tensor

$$\partial_\mu \langle S_{\mu\ldots}(x_1)\ldots\rangle = 0$$

Conformal symmetry or conservation can be solved independently, but so far not together

# Motivation

- conformal bootstrap: 4pt function of currents and $T_{\mu\nu}$



- new formalism for CFT correlators

  connection between CFT correlation functions and scattering amplitudes

# Embedding formalism

- conformal group $\mathrm{SO(d+1,1)}$ of $\mathbb{R}^{\mathrm{d}}$ acts linearly in the embedding space $\mathbb{R}^{\mathrm{d+1,1}}$
  - space of the light-like rays $p^2 = 0$ in $\mathbb{R}^{\mathrm{d+1,1}}$ is mapped into original $\mathbb{R}^{\mathrm{d}}$
  - auxiliary variables $z^\mu$ "take care" of spin indexes

    $\langle \mathcal{O}_{\mu\ldots}(x_1)\ldots\mathcal{O}_{\ldots\nu}(x_n)\rangle \leftrightarrow \mathcal{P}_{\mu\ldots\nu}(p_1,\ldots,p_n)$

    $P(z_1,\ldots,z_n,p_1,\ldots,p_n) = z_1^\mu \ldots z_n^\nu\,\mathcal{P}_{\mu\ldots\nu}(p_1,\ldots,p_n)$

  $P(z,p)$ must obey certain symmetries; all such $P(z,p)$ can be built as polynomials of $H_{ij}(z,p)$ and $V_{i,jk}(z,p)$ with coefficients depending on cross-ratios

- conformal correlators $P(z,p)$ in $\mathbb{R}^d$ are in one to one correspondence with scattering amplitudes in $\mathbb{R}^{d+1}$

  non-conserved operators / scattering of massive particles

# Solving constraints explicitly for 4pt function

- four point function of currents is parametrized by 43 functions of cross-ratios

$$
\begin{aligned}
\langle JJJJ \rangle &= \sum_I^{43} f_I(u,v)\,\mathcal{P}^I(z,x) \\
\langle \mathcal{O}^{\mathrm{d}} JJJ \rangle &= \sum_{\mathcal{I}}^{14} \mathsf{f}_{\mathcal{I}}(u,v)\,\mathcal{P}^{\mathcal{I}}(z,x)
\end{aligned}
$$

- conservation is a system of first order linear PDE

$$
\boxed{\left(A_I^{\mathcal{I}}(u,v) + B_I^{\mathcal{I}}(u,v)\frac{\partial}{\partial u} + C_I^{\mathcal{I}}(u,v)\frac{\partial}{\partial v}\right) f^I(u,v) = 0}
$$

matrices $A, B, C$ are $4\times\mathbf{14}\ \times\ \mathbf{43}$

- $\mathbb{Z}_2 \times Z_2$ permutations that respect cross-ratios

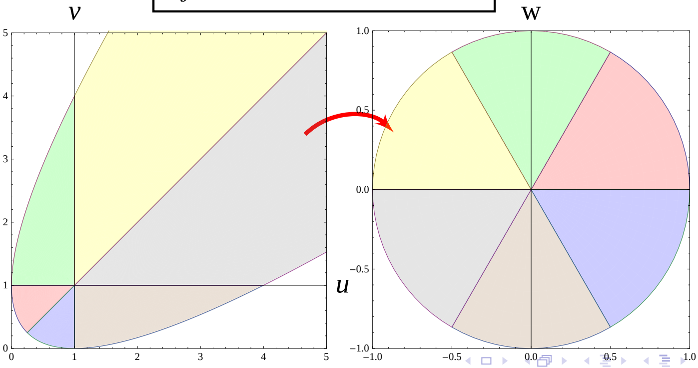
$$
\boxed{\Lambda^I_J(u,v) f^J(u,v) = f^I(u,v)}
$$

# Convenient coordinates for crossing-symmetry

- bose-symmetry acts on $\phi$ in a simple way but leaves $r$ invariant $\boxed{S_{12}: \phi \to -\phi \quad S_{13}: \phi \to 2\pi/3 - \phi}$

$$\boxed{u = \left|\frac{1+\lambda w}{1+w}\right|^2 \quad v = \left|\frac{1+\bar{\lambda} w}{1+w}\right|^2 \qquad w = re^{i\phi},\ \lambda = e^{2\pi i/3}}$$

$$\boxed{S^i_j(r,\phi) g^j(r,\phi) = g^i(r,\phi')}$$



# Cauchy problem and counting of DOF

- choosing one coordinate $t(u, v)$ as "time"

  "time" derivatives of only 12 $f^I$ out of 19 enter conservation equations

  $$\boxed{f^I = (\underbrace{h_1, \ldots, h_{12}}_{\text{constrained}}, \underbrace{g_1, \ldots, g_7}_{\text{unconstrained}})}$$

  12 equations

  $$\partial_t h = \ldots h + \ldots \partial_x h + \ldots g + \ldots \partial_x g$$

  2 constraints of first class

  $$\ldots h + \ldots \partial_x h = 0$$

- crossing symmetry $S_3$ permutes $g_i$ independently

  $(u, v) \to (v, u), \ (u/v, 1/v)$

  $$\boxed{S^i_j(u, v) g^j(u, v) = g^i(u', v')}$$

# Conformal bootstrap for four equivalent operators

## split into "unconstrained" and "dependent" DOF is not unique

$$\boxed{f^I = (\underbrace{h_1, \ldots, h_k}_{\text{constrained}}, \underbrace{g_1, \ldots, g_\ell}_{\text{unconstrained}}) \quad \Rightarrow \quad h[g]}$$

- $g_i$ satisfy crossing-symmetry in the "bulk" (all $r$)

$$S^i_j(r,\phi)g^j(r,\phi) = g^i(r,\phi')$$

- $h_\alpha$ satisfy crossing-symmetry at the boundary $r = r_0$

$$S^\alpha_\beta(r_0,\phi)h^\beta(r_0,\phi) = h^\alpha(r_0,\phi')$$

subject to other boundary constraints

Conjecture: boundary data is irrelevant for bootstrap, i.e. if correlation function admits conformal block decomposition $g$ fix $h$

# Regularity and redundancy of "boundary" data

## Conjecture: regular behavior at the boundary unambiguously fixes "boundary" data

ongoing work with P. Kravchuk

- example: $\langle J\mathcal{O}\mathcal{O}\mathcal{O}\rangle$

two functions, one equation

$z = x + iy$

$$g_-(x,y) = \frac{i}{y}\left(f(x) + \int_0^y \partial_x g_+(x,y) y dy\right)$$

regularity requires $f(x)$ to vanish

- $\langle JJJJ\rangle$: global solution can be obtained with an appropriate choice of coordinates

global solution is required to fix all boundary conditions

# Bootstrap: punchline

- crossing symmetry condition applies to a small number of "independent" functions $g_i$; all other DOF are ambiguously fixed
- the choice of "independent" DOF is not unique

| correlator | d=3 | d=4 | d=5 | d$\geq$ 6 |
|---|---|---|---|---|
| $\langle JJJJ \rangle$ | 5 | 7 | 7 | 7 |
| $\langle TTTT \rangle$ | 5 | 22 | 28 | 29 |

number of "independent" functions/families of conformal blocks relevant for corresponding parity-even 4pt functions in various dimensions

# Finding general solution

- zero curvature equation $\partial_\mu A_\nu - \partial_\nu A_\mu = 0 \Rightarrow A_\mu = \partial_\mu \alpha$

$$\text{example } \langle J\mathcal{O}\mathcal{O}\mathcal{O}\rangle: \quad g_\pm = \frac{\pm\partial_\pm \alpha}{z-\bar{z}}$$

- special solution/conformal blocks to satisfy Ward identities for charged operators

$$\partial_\mu \langle J_\mu(x)\ldots\rangle = \delta(x-y)\ldots$$

- system of differential equations $Df = 0$ can be solved in terms of compatibility operator $\mathcal{D}$, $f = \mathcal{D}\alpha$,

$$D\mathcal{D} = 0$$

there is an algorithmic way to construct $\mathcal{D}$; for $\langle JJJJ\rangle$ in $d = 3$ there are $5$ functions $\alpha$, as expected;

# New formalism?

## Relation between CFT correlators and scattering amplitudes

- scattering amplitudes for massless particles can be constructed in terms of $V, H$
- choice of compatibility operator $\mathcal{D}$ is not unique
  - for $\langle J_\mu \dots \rangle$ solution can be expressed as $\partial_\nu \langle F_{\mu\nu} \dots \rangle$
  - more generally $\tilde{\mathcal{D}}$ can be built of $z_{[m} p_n \frac{\partial}{\partial p^{k]}}$

$$D\tilde{\mathcal{D}} = 0$$

no clear understanding yet how to construct *all* conformal correlators that include conserved fields

# Conclusions

- conservation significantly reduces number of "independent" DOF which drastically simplifies (numerical) bootstrap; "boundary" data is fixed by regularity

- all set of Ward identities (including conservation) can be solved explicitly in terms of differential operators acting on independent functions; such a representation is not unique

- at this moment there is no universal mathematically elegant formalism to express conformal correlators in terms of unrestricted DOF that would make a connection with scattering amplitudes