

Comments on tensionless strings

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Aim:

understand tensionless string limit of $\text{AdS}_5/\text{CFT}_4$
– relation to HS theories, etc.

superstring in $AdS_5 \times S^5 \sim SU(N) \mathcal{N} = 4$ SYM

string parameters: tension T and string coupling g_s

$$T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}, \quad g_s = g_{\text{YM}}^2 = \frac{\lambda}{N}, \quad \lambda = g_{\text{YM}}^2 N$$

't Hooft limit: large N , fixed λ

$\lambda \rightarrow 0$: “zero-tension” limit $T \rightarrow 0$ is subtle

- does not mean $R \rightarrow 0$ or renormalization of string tension [would contradict what is now known about duality from integrability, supersymmetry and localization – exact BMN dispersion relation, exact results for BPS Wilson loops, Konishi operator anomalous dimension]

- should correspond to theory in AdS_5 : free SYM is CFT – should be dual to massless+ massive higher-spin theory in AdS_5

- $T \rightarrow 0$ is strong-coupling limit in world sheet theory:
should be taken in quantum string theory –
start with exact string spectrum in $AdS_5 \times S^5$ for fixed λ ,
then take $\lambda \rightarrow 0$ in global AdS energy
→ match dimensions of primary operators in free SYM CFT
- same for correlation functions:
computed in AdS, finite in $\lambda \rightarrow 0$,
overall coefficients controlled by $G_N \sim N^{-2}$
(modulo normalization to account for
spectrum degeneracy in $T \rightarrow 0$ limit)
- still: attempt to take $T \rightarrow 0$ directly in string action?
need to fix some charge – e.g. l.c. gauge momentum
 $P^+ = \sqrt{\lambda} p^+ = \text{fixed}$ or $J = S^5$ angular momentum

- $AdS_5 \times S^5$ l.c. gauge Lagrangian [Metsaev, Thorn, AT 02]

$$\mathcal{L} = P^+ \left[\dot{x}_\perp^2 + (\dot{Z}^M - i\eta_i \rho^{MNi}{}_j \eta^j Z_N Z^{-2})^2 + i(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i - h.c.) \right. \\ \left. - Z^{-2}(\eta^2)^2 - T^2 Z^{-4} (p^+)^{-2} (x'_\perp{}^2 + Z'^M Z'^M) \right] \\ - T \left[|Z|^{-3} \eta^i \rho_{ij}^M Z^M (\theta'^j - i\sqrt{2}|Z|^{-1} \eta^j x') + h.c. \right]$$

$x_\perp = x_1 + ix_2$ transverse coordinates of Poincare patch

$$ds^2 = Z^{-2} (dx_m dx_m + dZ_M dZ_M), \quad Z^M = Z n_M, \quad n_M n_M = 1$$

- $T \rightarrow 0$: drop all σ derivatives in $I = \int d\tau \int d\sigma \mathcal{L}$

parameter $P^+ = \sqrt{\lambda} p^+$ plays role of \hbar^{-1} [AT 02]

$$I_{T \rightarrow 0} = P^+ \int d\tau \int d\sigma \left[\dot{x}_\perp^2 + i(\theta^i \dot{\theta}_i + \theta_i \dot{\theta}^i) + i(\eta^i \dot{\eta}_i + \eta_i \dot{\eta}^i) \right. \\ \left. + (\dot{Z}^M - i\eta_i \rho^{MNi}{}_j \eta^j Z_N Z^{-2})^2 - Z^{-2}(\eta^2)^2 \right]$$

- $\lambda = 0$ action describes collection of particles (“string bits”) moving in AdS: integrable classical dynamics (geodesics)
- no ∂_σ : huge degeneracy in spectrum – seen e.g. in pp-wave case [Metsaev, AT 02; Lindstrom, Wulff et al 04] leads to divergence in free partition function and correlators: divide by gauge volume or consider ratios of corr. functions
- should be reflecting new gauge symmetry at $\lambda = 0$ in SYM: ∞ set of conserved HS currents \rightarrow massless HS fields in AdS [Sundborg 00; Witten 00]
- spectrum: “leading” Regge trajectory of massless HS fields + higher trajectories of massive fields in AdS
- massless HS subsector: AdS_5 Vasiliev-type theory dual to bilinear cons. currents $J_s \sim \text{tr}(\Phi \partial^s \Phi)$, $\text{tr}(F_{mn} \partial^s F_{mn})$, etc.
- infinite set of extra massive fields dual to “long” SYM ops: $n > 2$ free fields $\mathcal{O}_{n,s} \sim \text{tr}(\Phi \partial^{s_1} \Phi \dots \partial^{s_2} \Phi)$

- symbolic action of AdS dual for adjoint free field CFT:

$$S = N^2 \int \sum [\phi_s(\nabla^2 + \dots)\phi_s + \phi_s^3 + \dots + \psi_{n,s}(\nabla^2 + M_{n,s}^2)\psi_{n,s} + \dots]$$

$\psi_{n,s}$ – massive fields dual operators $\mathcal{O}_{n,s}$ with n fields

- puzzling feature: for $N = \infty$ have any $n < N = \infty$

but for finite N number of elementary fields is finite $n < N$:

trace factorizes (get multi-particle states) –

finite no. of “Regge trajectories”

- the coupling $1/N^2$ is not just as an overall Planck constant?

local action description is not appropriate unless $N = \infty$?

- similar simpler model – adjoint $U(N)$ scalar CFT_d

AdS_{d+1} theory: massless HS sector + ∞ set of massive fields;

can be also described as “zero-tension” limit

of some bosonic theory in AdS_{d+1} (no critical dim for $T = 0$)?

- even simpler model: vectorial AdS/CFT – no massive states

Taking zero-tension limit:

alternative: static or l.c. gauge adapted to BMN vacuum

$t = \tau$ and momentum along $S^1 \subset S^5$ fixed: $p_\varphi = \mathcal{J}$

(i) for fixed $J = \sqrt{\lambda} \mathcal{J}$ can take $T = \frac{\sqrt{\lambda}}{2\pi} \rightarrow 0$ limit

– dropping all σ -derivatives;

(ii) only remaining parameter is overall J factor

e.g. bosonic part of string action in $R_t \times S^n$

$$L_S = -\frac{1}{2} \left[G(y) \partial^a \varphi \partial_a \varphi + F(y) \partial^a y_s \partial_a y_s \right]$$
$$G = \frac{(1 - \frac{1}{4} y_s^2)^2}{(1 + \frac{1}{4} y_s^2)^2}, \quad F = \frac{1}{(1 + \frac{1}{4} y_s^2)^2}$$

to fix $p_\varphi = \mathcal{J}$ gauge apply first T-duality $\varphi \rightarrow \tilde{\varphi}$

and then fix static gauge $t = \kappa\tau$, $\tilde{\varphi} = \mathcal{J}\sigma$

$$I = -\sqrt{\lambda} \int d\tau \int \frac{d\sigma}{2\pi} \sqrt{h}$$

$$h = [1 - F(y)\dot{y}_r^2] [\mathcal{J}^2 G^{-1}(y) + F(y)y_s'^2] + [F(y)\dot{y}_r y_r']^2$$

set $J = \sqrt{\lambda} \mathcal{J}$

$$I = -J \int d\tau \int \frac{d\sigma}{2\pi} \sqrt{[1 - F(y)\dot{y}_r^2] [G^{-1}(y) + \frac{\lambda}{J^2} F(y)y_s'^2] + \frac{\lambda}{J^2} [F(y)\dot{y}_r y_r']^2}$$

• zero tension limit: $\lambda \rightarrow 0$ for fixed J , i.e. $\mathcal{J} = \frac{J}{\sqrt{\lambda}} \rightarrow \infty$

removes all σ -derivative terms

$$\begin{aligned} I_0 &= -J \int d\tau \int \frac{d\sigma}{2\pi} \sqrt{G^{-1}(y) [1 - F(y)\dot{y}_r^2]} \\ &= -J \int d\tau \int \frac{d\sigma}{2\pi} \sqrt{\frac{(1 + \frac{1}{4}y_r^2)^2 - \dot{y}_r^2}{(1 - \frac{1}{4}y_r^2)^2}} \end{aligned}$$

- $J \gg 1$ corresponds to semiclassical expansion expanding in powers of $\tilde{y} = J^{1/2}y$:

$$I_0 = \frac{1}{2} \int d\tau \int \frac{d\sigma}{2\pi} [\dot{\tilde{y}}_r^2 - \tilde{y}_r^2 + O(J^{-1}\tilde{y}^4)]$$

8+8 massive modes in $AdS_5 \times S^5$ case

- same in flat space: $F = G = 1$

$$I_0 = -J \int d\tau \int \frac{d\sigma}{2\pi} \sqrt{1 - \dot{y}_r^2}$$

collection of free particles

[tensionless limit in flat space is not defined unless one fixes one momentum]

Conformal symmetry in zero-tension limit?

- flat space: $\alpha' \rightarrow \infty$, no scale, massless higher spins
 - conformal invariance? Why?
- previous suggestion [Lindstrom, Sundborg, Theodoris 91]
 $T\sqrt{g}g^{ab}$ degenerate in the limit: can be replaced by V^aV^b

$$I = \int d^2z V^a V^b \partial_a X^m \partial_b X^n G_{mn}(X)$$

$V^a(z)$ auxiliary vector density –
target space Weyl invariance?

sp-time conformal group in flat case?

- apparently not: $V^a(z)$ is not same as $V^a(x(z))$

that is required to compensate for conformal transformations

indeed: standard massless HS fields are not conformal for $s > 1$

Galilean conformal symmetry?

$$S = T \int d^2\xi \sqrt{-\det \gamma_{ab}}, \quad \gamma_{ab} = \partial_a X^m \partial_b X^n \eta_{mn}$$

generalised momenta satisfy

$$P^2 + T^2 \gamma \gamma^{00} = 0, \quad P_m \partial_\sigma X^m = 0$$

add with Lagrange multipliers and integrate out momenta

$$S = \int d^2\xi \frac{1}{2\lambda} [\dot{X}^2 - 2\rho \dot{X}^m \partial_\sigma X_m + \rho^2 \partial_\sigma X^m \partial_\sigma X_m - 4\lambda^2 T^2 \gamma \gamma^{00}]$$

$$S = -\frac{1}{2} T \int d^2\xi \sqrt{-g} g^{ab} \partial_a X^m \partial_b X^n \eta_{mn}$$

$$g^{ab} = \begin{pmatrix} -1 & \rho \\ \rho & -\rho^2 + 4\lambda^2 T^2 \end{pmatrix}$$

tensionless limit: replace degenerate metric g^{ab} by $V^a V^b$

$$V^a = \frac{1}{\sqrt{2\lambda}}(1, \rho)$$

$$S_{T \rightarrow 0} = \int d^2\xi V^a V^b \partial_a X^m \partial_b X^n \eta_{mn}$$

Residual symmetries:

under $\xi^a \rightarrow \xi^a + \epsilon^a$ vector density V^a transforms as:

$$\delta V^a = -V^\beta \partial_b \epsilon^a + \epsilon^b \partial_b V^a + \frac{1}{2} V^a \partial_b \epsilon^b$$

gauge:

$$V^a = (v, 0) \rightarrow S_{T \rightarrow 0} = \int d^2\xi v \dot{X}^m \dot{X}^n \eta_{mn}$$

residual symmetry (analog of Virasoro): [Bagchi 14]

$$\epsilon^a = (f'(\sigma)\tau + g(\sigma), f(\sigma))$$

$$\delta F = [f'(\sigma)\tau\partial_\tau + f(\sigma)\partial_\sigma + g(\sigma)\partial_\tau]F = [L(f) + M(g)]F$$

- generators:

$$L(f) = f'(\sigma)\tau\partial_\tau + f(\sigma)\partial_\sigma, \quad M(g) = g(\sigma)\partial_\tau$$

$$[L(f_1), L(f_2)] = L(f_1f_2' - f_1'f_2), \quad [L(f), M(g)] = M(fg' - f'g)$$

$$[M(g_1), M(g_2)] = 0$$

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [L_m, M_n] = (m-n)M_{m+n}, \quad [M_m, M_n] = 0$$

same as 2d Galilean Conformal Algebra

- Virasoro symmetry is replaced in “conformal gauge” by 2d Galilean conformal symmetry

$$L_n = ie^{in\sigma}(\partial_\sigma + in\tau\partial_\tau), \quad M_n = ie^{in\sigma}\partial_\tau$$

- GCA in any d : conjectured symmetry of Galilean CFT: non-relativistic analog conformal symmetry (but ∞ -dimensional)

- infinite dim symmetry in any flat d ($i = 1, \dots, d - 1$)

$$L_n = t^{n+1} \phi_t + (n-1)t^n x_i \partial_i, \quad M_n^i = t^{n+1} \partial_i, \quad J_{ij}^n = t^n (x_i \partial_j - x_j \partial_i)$$

$$[L_n, L_m] = (n-m)L_{n+m}, \quad [L_n, M_m^i] = (n-m)M_{n+m}^i, \quad [M_n^i, M_m^j] = 0$$

- finite dim subgroup is contraction of relativistic conf algebra

$$L_{-1,0,+1} = H, D, K^0 \quad M_{-1,0,+1}^i = P^i, B^i, K^i$$

K^0, K^i special conformal and B^i Galilean boosts

- quantum version of the GCA in $d = 2$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}c_L(n^3 - n)\delta_{n+m,0},$$

$$[L_n, M_m] = (n-m)M_{n+m} + \frac{1}{12}c_M(n^3 - n)\delta_{n+m,0}, \quad [M_n, M_m] = 0.$$

contraction of two copies of the Virasoro algebra $\mathcal{L}_n, \bar{\mathcal{L}}_n$

$$\mathcal{L}_n + \bar{\mathcal{L}}_n = L_n, \quad \mathcal{L}_n - \bar{\mathcal{L}}_n = \frac{1}{\epsilon}M_n, \quad c + \bar{c} = c_L, \quad c - \bar{c} = \frac{1}{\epsilon}c_M$$

if $c = \bar{c}$ then in the limit $c_L = c_M = 0$

no anomalies in Virasoro \rightarrow no anomalies in GCA

cf. no critical dimension for tensionless string [Lizzi et al 86]

- is this symmetry really fundamental for $T = 0$ string?...

Virasoro is residual gauge symmetry of tensile string;

same should be for this symmetry

- but unlikely it is actually responsible for

degeneracy of spectrum in $T \rightarrow 0$ limit

in non-trivial curved space case

Superstring in pp-wave limit of $AdS_5 \times S^5$ [Metsaev 02]

$$ds^2 = dx^+ dx^- - f^2 x_I^2 dx^+ dx^+ + dx^I dx^I$$

$$F_{+1234} = F_{+5678} = f = \text{curv. scale}, \quad I = 1, \dots, 8$$

- l.c. gauge: $x^+ = p^+ \tau$, $\Gamma^+ \theta^{II} = 0$

$$L = \partial^a x^I \partial_a x^I - m^2 x^I x^I + \text{fermions}$$

- eqs of motion: 8 massive bosons + 8 massive fermions

$$\omega_n = \sqrt{k_n^2 + m^2}, \quad k_n = 2\pi n, \quad m = 2\pi\alpha' p^+ f = 2\pi\mu$$

- Hamiltonian $H = P^-$

$$H = f(a_0^I \bar{a}_0^I + 2\bar{\theta}_0 \bar{\gamma}^- \Pi \theta_0 + 4) \\ + \frac{1}{\alpha' p^+} \sum_{\mathcal{I}=1,2} \sum_{n=1}^{\infty} \sqrt{n^2 + (\alpha' p^+ f)^2} (a_n^{\mathcal{II}} \bar{a}_n^{\mathcal{II}} + \eta_n^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_n^{\mathcal{I}})$$

Connection to BMN limit of $AdS_5 \times S^5$ ($R = 1$)

$$P^+ = J = \sqrt{\lambda} p^+$$

$$(E - J)J = \sum_n \sqrt{J^2 + \lambda n^2}, \quad E - J = \sum_n \sqrt{1 + \frac{\lambda}{J^2} n^2}$$

$$\mathcal{J} = p^+ = \frac{J}{\sqrt{\lambda}} \rightarrow \infty$$

- flat space limit is $f \rightarrow 0$: $P^- p^+ = (\text{mass})^2$
- dimensionless parameters: $m = 2\pi\alpha' p^+ f \equiv 2\pi\mu$ and $\alpha'(p^+)^2$
- zero-tension limit: $\alpha' p^+ f \rightarrow \infty$ [Metsaev, AT 02]

$$H_0 = f \left[(a_0^I \bar{a}_0^I + 2\bar{\theta}_0 \bar{\gamma}^- \Pi \theta_0 + 4) + \sum_{\mathcal{I}=1,2} \sum_{n=1}^{\infty} (a_n^{\mathcal{I}I} \bar{a}_n^{\mathcal{I}I} + \eta_n^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_n^{\mathcal{I}}) \right]$$

- vacuum = product of zero-mode vac and Fock oscillator vac
 $\bar{a}_0^I |0\rangle = 0$, $\bar{\theta}_0^\alpha |0\rangle = 0$, $\bar{a}_n^{\mathcal{I}I} |0\rangle = 0$, $\bar{\eta}_n^{\mathcal{I}\alpha} |0\rangle = 0$, $n = 1, 2, \dots$
- generic state: $|\Phi\rangle = \Phi(a_0, a_n, \theta_0, \eta_n) |0\rangle$

subspace of physical states $N^1 |\Phi_{phys}\rangle = N^2 |\Phi_{phys}\rangle$

$$N^{\mathcal{I}} = \sum_{n=1}^{\infty} k_n (a_n^{\mathcal{I}I} \bar{a}_n^{\mathcal{I}I} + \eta_n^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_n^{\mathcal{I}})$$

- large degeneracy of states in energy

Degeneracy of states in 0-tension limit leads to divergences:

- flat space: $M^2 = \frac{1}{\alpha'} N \rightarrow 0$ – appearance of massless fields
- partition function becomes divergent:

integral over longitudinal directions is no longer suppressed
producing volume of the gauge group that appears in the limit

- analogy: massive \rightarrow massless vector

$$Z = \int [dA] \exp[-\int d^4x (F_{ab}F^{ab} + m^2 A_a A^a)]$$

$$A_a = A_a^\perp + \partial_a \phi$$

$$F_{ab}F^{ab} + m^2 A_a A^a = A_a^\perp (-\partial^2 + m^2) A_a^\perp + m^2 \partial_a \phi \partial^a \phi$$

if $m \rightarrow 0$ integral over ϕ is no longer suppressed: jump in d.o.f.

$\int [d\phi]$ is volume of gauge group that appears in the limit $m \rightarrow 0$

one needs to divide over $\int [d\phi]$ to get finite partition function

still *ratios* of *certain* correlation functions have smooth limit

e.g.
$$\frac{\langle F_{mn} F_{kl} \dots F_{pq} \rangle}{\langle F_{mn} F_{kl} \rangle}$$

Partition function in pp-wave background

8b+8f: free energy on $R_L \times S^1_\beta \times R^8$ [Grignani et al 03]

$$F_b = \frac{1}{\beta} \text{Tr} \ln(1 - e^{-\beta p^0}) = - \sum_{k=1}^{\infty} \frac{1}{k\beta} \text{Tr} e^{-\frac{k\beta}{2}(p^+ - p^-)}$$

$$F = F_b + F_f = - \sum_{k=1}^{\infty} \frac{1 - (-1)^k}{2k\beta} \text{Tr} e^{-\frac{k\beta}{2}(p^+ - p^-)}$$

$$F = - \frac{L}{4\pi^2 \alpha'} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \sum_{k=1,3,5,\dots}^{\infty} e^{-\frac{k^2 \beta^2}{4\pi \alpha' \tau_2}} G(\tau_1, \tau_2) - \frac{\pi L}{24\beta^2}$$

$$G(\tau_1, \tau_2) \equiv \prod_{n=-\infty}^{\infty} \left[\frac{1 + \exp(-2\pi\tau_2 \sqrt{n^2 + \mu^2} + 2\pi i\tau_1 n)}{1 - \exp(-2\pi\tau_2 \sqrt{n^2 + \mu^2} + 2\pi i\tau_1 n)} \right]^8$$

$L \rightarrow \infty$ = length in longitudinal 9-th direction, $\mu = \alpha' p^+$

• flat-space limit: $\mu = 0$

• naive $\mu \rightarrow \infty$ limit: $G \rightarrow 1$, $F = -\frac{\pi L}{6\beta^2}$:

free energy density of gas of massless particles in 2d

- zero-tension limit: $\mu \rightarrow \infty$ with scale f fixed:
 $\alpha' p^+ \rightarrow \infty$ while $\beta^2 p^+ = \text{fixed}$, $L p^+ = \text{fixed}$

$$G = \prod_{n=-\infty}^{\infty} \left(\frac{1 + e^{-2\pi\tau'_2 + 2\pi i\tau_1 n}}{1 - e^{-2\pi\tau'_2 + 2\pi i\tau_1 n}} \right)^8, \quad \tau'_2 = \mu\tau_2$$

product is divergent: reflects degeneracy in the 0-tension limit

- interpretation: new gauge symmetry appearing in $T = 0$ limit
 - divide over its volume to define partition function?
- consider ratios of correlation functions?

Interactions: 3-point function [Klebanov, Spradlin, Volovich 02]

$$|V\rangle = \exp \left[\frac{1}{2} \sum_{r,s=1}^3 \sum_{m,n=-\infty}^{\infty} a_{m(r)}^{I\dagger} \overline{N}_{mn}^{(rs)} a_{n(s)}^{J\dagger} \delta_{IJ} \right] |0\rangle$$

Neumann matrices for $m, n > 0$

$$\overline{N}_{mn}^{(rs)} = \delta^{rs} \delta_{mn} - \left[C_{(r)}^{1/2} C^{-1/2} A^{(r)\text{T}} \Gamma_+^{-1} A^{(s)} C^{-1/2} C_{(s)}^{1/2} \right]_{mn}$$

$$A_{mn}^{(1)} = (-1)^{m+n+1} \frac{\sqrt{mn}}{\pi} \frac{y \sin(\pi my)}{n^2 - m^2 y^2}$$

$$A_{mn}^{(2)} = (-1)^m \frac{\sqrt{mn}}{\pi} \frac{(1-y) \sin(\pi my)}{n^2 - m^2 (1-y)^2}$$

$$A_{mn}^{(3)} = \delta_{mn}, \quad C_{mn} = m \delta_{mn},$$

$$C_{mn}^{(2)} = \delta_{mn} \sqrt{m^2 + \mu^2 (1-y)^2},$$

$$B_m = \frac{(-1)^{m+1} \sin(\pi my)}{\pi y (1-y) \alpha' p^+ m^{3/2}}$$

$$C_{mn}^{(1)} = \delta_{mn} \sqrt{m^2 + \mu^2 y^2}$$

$$C_{mn}^{(3)} = \delta_{mn} \sqrt{m^2 + \mu^2}$$

- $\mu \rightarrow \infty$: simplifications
and divergences due to degeneracy

Open questions

- which is efficient description of $T = 0$ limit?

is there a constructive definition of (string ?) theory directly in the limit?

or only makes sense as limit of exact quantum spectrum ?

- new gauge symmetries of string theory in $T = 0$ limit?
- quantum bosonic $T = 0$ string theory well-defined in AdS_{d+1} ?
dual to free scalar adjoint CFT_d ?
- precise connection to massless HS theory in AdS ?
- how to reproduce e.g. spectrum and simplets correlators of primary operators in $g_{\text{YM}} = 0$ YM $SU(N)$ theory from tensionless string in AdS ?
- generalization to superstring in AdS dual to SYM case