Conformal gauge fields as boundary values

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Based on:

A. Chekmenev, M.G. - to appear

Also:

X. Bekaert, M.G., arXiv:1305.0162, arXiv:1207.3439

G. Barnich, X. Bekaert, M.G 1506.00595

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- AdS/CFT heavily relies on the notion of boundary values.

- Standard approach: Lagrangian for conformal fields arise as log-divergent term in the on-shell action:

$$S[\phi_0] = \int d\rho d^d x \sqrt{g} S^{bulk}[\phi[\phi_0]] , \qquad \frac{1}{\rho^{\Delta_-/2}} \phi \Big|_{z=0} = \phi_0$$

- Alternative approach: conformal equations can be seen as obstructions of extending boundary value to a bulk solution. Can be formulated entirely at EOMs level. *Fefferman, Graham*

  Gauge covariant version for free fields *X. Bekaert, M.G. (2012),(2013)*

- It is tempting to say that all conformal (gauge) fields can be seen as boundary values of the respective AdS gauge fields and both have to be described in a unified framework where the fundamental field lives on the ambient space.

Plan

- Review of the ambient space approach to boundary values.

- Mixed symmetry gauge fields on AdS and their boundary values

- Manifestly conformal ambient formulation of conformal gauge fields

- Conclusion

AdS scalar

AdS_{d+1} space

$$ds^2 = \frac{1}{4\rho^2}(d\rho)^2 + \frac{1}{\rho}(\eta_{ab}dx^a dx^b)\,, \qquad (a, b = 0, \dots, d-1)$$

Boundary: $\rho = 0$

Massive scalar

$$(\nabla^2 - m^2)\varphi = 0$$

Two $\mathfrak{o}(d,2)$-invariant choices of asymptotic behavior:

$$\varphi(x,\rho) = \rho^{\Delta_\pm/2}\phi_\pm(x,\rho)\,, \qquad (\phi_0)_\pm = \phi_\pm|_{\rho=0} \;\; - \;\; \text{boundary value}$$

Two solutions $\Delta = \Delta_\pm$, $\Delta_+ \geqslant \Delta_-$ to

$$m^2 = \Delta(\Delta - d) \quad \Leftrightarrow \quad \Delta_\pm = \frac{1}{2}(d \pm \sqrt{d^2 + 4m^2}) \qquad \Delta_+ + \Delta_- = d$$

Ambient space

$\mathbb{R}^{d,2}$ where $o(d,2)$ acts by infinitesimal isometries. *Dirac (1935)*

X^A ($A = +, -, 0, 1, 2, \ldots, d-1$) coordinates on $\mathbb{R}^{d,2}$ where

$\eta_{+-} = 1 = \eta_{-+}, \qquad \eta_{ab} = \text{diag}(-1, +1, \ldots, +1) \qquad a, b = 0, 1, 2, \ldots, d-1$

Useful notations $X \cdot Y = \eta_{AB} X^A Y^B$ and $X^2 = X \cdot X$.

AdS_{d+1} space $X^2 = -1$. Explicit embedding:

$$X^+ = \rho^{-\frac{1}{2}}, \quad X^- = -\frac{1}{2}(\rho + x^a x_a)\,\rho^{-\frac{1}{2}}, \quad X^a = \rho^{-\frac{1}{2}} x^a$$

Conformal space $Conf_d$: quotient of hyper-cone $X^2 = 0$ by equivalence relation $X \sim \lambda X, \quad \lambda \in \mathbb{R}\backslash 0$. Can be seen as a surface (Minkowski metric)

$$X^2 = 0\,, \qquad X^+ = 1$$

Ambient scalar

Ambient lift of AdS scalar:

$$\left(X \cdot \frac{\partial}{\partial X} + \Delta_{\pm}\right)\Phi = 0\,, \qquad \Box\Phi = 0 \qquad\qquad (*)$$

For Φ defined for $X^2 < 0$ this is equivalent to $(\nabla^2 - m^2)\varphi = 0$ with $\varphi = \Phi|_{X^2=-1}$. In this domain $\Delta_{\pm}$ descriptions are equivalent.

Boundary value: $\Phi|_{X^2=0}$, i.e. value at the hyper-cone (may be ill-defined).

$\Delta = \Delta_{+}$ – subleading b. value

$\Delta = \Delta_{-}$ – leading b. value

To work in e.g. Minkowski space $\Phi|_{X^2=0,X^+=1}$.

The same system on the entire $\mathbb{R}^{d,2}\backslash\{0\}$:

$$\left(X\cdot\frac{\partial}{\partial X}+\Delta\right)\Phi=0\,,\qquad \Box\Phi=0 \qquad (*)$$

For Δ generic or $\Delta=\Delta_+$ it simply describes the respective boundary value as a conformal field in a manifestly conformal way.

For $\Delta=\Delta_-=\frac{d}{2}-\ell$, $\ell=1,2,\ldots$ the leading boundary value is constrained. More precisely, $\phi_0(x)=\Phi|_{X^2=0,X^+=1}$ is subject to

$$(\Box_0)^\ell\phi_0=0\,,\qquad \Box_0=\frac{\partial}{\partial x^a}\frac{\partial}{\partial x_a}$$

In addition, $(*)$ describes a subspace of subleading solutions:

$$\Phi=(X^2)^\ell\alpha\,,\qquad \left(X\cdot\frac{\partial}{\partial X}+\Delta_+\right)\alpha=0\,,\qquad \Box\alpha=0$$

The subspace is isomorphic to subleading solutions. The ambient system

$$\left(X\cdot\frac{\partial}{\partial X}+\Delta\right)\Phi=0\,,\qquad \Box\Phi=0\,,\qquad \Phi\sim\Phi+(X^2)^\ell\alpha\,.$$

The gauge equivalence eliminates the subleading. Equivalent to $\Box_0^\ell\phi_0=0$ on the boundary. Byproduct: manifestly $o(d,2)$ description of $\Box_0^\ell\phi_0=0$

This picture generalizes smoothly to the case of gauge fields. However,

- Take into account gauge invariance
- Tensor fields. Coordinate independent prescription for boundary behavior.

These can be resolved using the approach based on (BRST) first-quantized constrained systems in the ambient space.

The generalization to the case of totally symmetric fields

X. Bekaert, M.G. (2012), (2013)

Mixed symmetry fields. New feature – essentially reducible gauge theory.

The above approach differs from the usual one, where

$$\varphi(x,\rho) = \rho^{\Delta_-/2}\phi(x,\rho) = \rho^{\Delta_-/2}\Big(\phi_0(x) + \rho\,\phi_1(x) + \dots\Big) + \\ + \rho^{\Delta_+/2}\Big(\log\rho\,\phi_\ell(x) + \alpha(x) + \rho\,\psi_{\ell+1}(x) + \dots\Big).$$

log-term related to holographic anomaly *Heningson, Skenderis, 1998*.

Note that ϕ is not smooth at $\rho = 0$.

In this approach the extension from the boundary is not obstructed. The conformal action $\int d^dx\phi_0(\Box_0)^\ell\phi_0$ emerges as a log-divergent part of the bulk action evaluated on the solution with fixed boundary value.

In the ambient approach $\rho = 0$ at the hypercone $X^2 = 0$ and it is natural to require Φ smooth there.

$o(d,2)$-modules associated to ambient system

Keep using scalar as a toy model.

A system of PDE is entirely determined by its jet-space stationary surface which is equipped with total derivative operator *Vinogradov (1978)*

Closely related to unfolded formalism *Vasiliev (1988),...* .

Example: Minkowski scalar, jet space and total derivative

$$x^\mu, \quad \phi, \quad \phi_\mu, \quad \phi_{\mu\nu}, \dots , \qquad \partial^T = \frac{\partial}{\partial x^\mu} + \phi_\mu \frac{\partial}{\partial \phi} + \dots$$

The stationary surface is singled out by

$$\eta^{\mu\nu} \phi_{\mu\nu\dots} = 0$$

and can be coordinatized by

$$x^\mu, \quad \phi', \quad \phi'_\mu, \quad \phi'_{\mu\nu}, \dots , \qquad \eta^{\mu\nu} \phi'_{\mu\nu\dots} = 0$$

∂^T_μ is well defined on the surface.

For a linear system the stationary surface is a bundle over space time whose fibers are vector spaces $W(x)$. The space $W(x)$ can be visualized as a space of solutions of the PDE in the space of formal power series around a point x. For PDE of the form $T_i^a(x, \frac{\partial}{\partial x})\phi^i(x) = 0$ then $W(x)$ is defined as

$$T_i^a(x + y, \frac{\partial}{\partial y})\phi^i(y) = 0 \qquad x \text{ is a parameter here!}$$

If the system is invariant under group G, space $W(x)$ is a G-module. In general, $W(x)$ are not isomorphic for different x. However, if G acts transitively on the spacetime $W(x)$ is the same everywhere. In this case it is known as Weyl module in the unfolded formalism.

In general to each equivalence class of G-orbits in spacetime there is an associated module W.

Applying to the ambient system: $W(AdS_{d+1})$ is in general not isomorphic to $W(Conf_d)$. Instructive to compare these two for the ambient scalar. $X^2 = -1$, Convenient choice of the point of AdS_{d+1}: $X^{d+1} = 1, X^m = 0$. Adapted coordinates $y^m, \bar{y}$. Defining equations:

$$(y^m \frac{\partial}{\partial y^m} + (\bar{y}+1)\frac{\partial}{\partial \bar{y}})\Phi(y,\bar{y}) = 0\,, \qquad \frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m} - \frac{\partial}{\partial \bar{y}}\frac{\partial}{\partial \bar{y}}\Phi(y,\bar{y}) = 0$$

Solutions can be parameterized by $\phi(y)$ satisfying $\frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m}\phi = 0$. The $o(d,2)$-generators

$$J_{AB} = (X_A + Y_A)\frac{\partial}{\partial Y^B} - (X_B + Y_B)\frac{\partial}{\partial Y^A}$$

act as

$$J_{mn} = y_m \frac{\partial}{\partial y^n} - y_n \frac{\partial}{\partial y^m}$$

$$\widehat{P}_n \phi = \frac{\partial}{\partial y^n}\phi - \Pi\left(y^n \frac{(N+\Delta)(N+d-\Delta)}{d+1+2N}\phi\right), \qquad N = y^m \frac{\partial}{\partial y^m}$$

Note that the expression is unchaged under $\Delta \to d-\Delta$. The module was explicitly described *Shaynkman, Vasiliev (2003)*

$X^2 = 0$, Convenient choice $X^+ = 1, X^- = X^A = 0$. Defining equations:

$$(y^a \frac{\partial}{\partial y^a} + (y^+ + 1)\frac{\partial}{\partial y^+})\Phi(y,\bar{y}) = 0\,, \qquad \frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m} - 2\frac{\partial}{\partial y^+}\frac{\partial}{\partial y^-}\Phi(y,\bar{y}) = 0$$

Assuming for simplicity $\Delta \neq 0, -1, -2, \ldots$ solutions are parameterized by unconstrained $\phi(y^a)$. $o(d,2)$ acts as

$$J_{ab}\phi = \left(y_a \frac{\partial}{\partial y^b} - y_b \frac{\partial}{\partial y^a}\right)\phi,$$

$$D = J_{+-}\phi = (N + \Delta)\,\phi,$$

$$K_a = J_{+a}\phi = y_a\,(N + \Delta)\,\phi,$$

$$P_a = J_{-a}\phi = \left(\frac{\partial}{\partial y^a} - y_a \frac{1}{2(N + \Delta + 1)}\frac{\partial}{\partial y_c}\frac{\partial}{\partial y^c}\right)\phi.$$

This is generalized Verma module becuase K_a act freely. This is precisely the one encoding polywave equation *Shaynkman, Tipunin, Vasiliev (2003)*

Not surprisingly: singular vector is precisely the subleading.

Byproduct: Interesting oscilator realization of such modules.

The difference between two modules illustrates how the ambient approach to boundary values work: it is in general not clear how to obtain module $W(X^2 = 0)$ out of $W(X^2 = -1)$ in "intrinsic" terms. In the ambient terms it simply amounts to replacing X^A satisfying $X^2 = -1$ with that satisfying $X^2 = 0$.

In a more elaborated version X^A is promoted to the compensator field. A version of: *MacDowell, Mansouri (1977) Stelle, West (1979)*

Interesting alternative: unfolded approach *Vasiliev (2012)*

Mixed symmetry ambient tensors

Generating functions depending on X^A and P_i^A $1 = 1, \dots n-1$

$$\Phi(X,P) = \Phi(X, P_1, \dots P_{n-1})$$

$o(d,2)$ algebra acts by

$$J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} + \sum_{i=1}^{n-1} (P_{iA} \frac{\partial}{\partial P_i^B} - P_{iB} \frac{\partial}{\partial P_i^A})$$

It is centralized by $sp(2n)$ (Dual in the sense of *Howe*)

$$\Box = \frac{1}{2}\partial^X \cdot \partial_X, \qquad \bar{\Box} = \frac{1}{2}X^2, \qquad X \cdot \partial_X + \frac{d+2}{2}$$

$$\partial_P^i \cdot \partial_X, \qquad P_i \cdot X, \qquad P_i \cdot \partial_X, \qquad X \cdot \partial_P^i,$$

$$\partial_P^i \cdot \partial_P^j, \qquad P_i \cdot P_j, \qquad P_i \cdot \partial_P^j + \delta_i^j \frac{d+2}{2}$$

Ambient description of mixed symmetry fields

Restrict to unitary fields *Metsaev (1995); Alkalaev, Shaynkman, Vasiliev (2003)*

Further developments *Skvortsov (2009), Boulanger, Iazeolla, Sundell (2009)*

Here we follow *Alkalaev, M.G. (2009,2011)*

Configurations of a unitary massless field of spin $\{s_1, s_2, \dots, s_{n-1}\}$ (it is assumed $s_1 \geq s_2 \geq \dots \geq s_{n-1}$ and $n-1 \leq [\frac{d}{2}]$) are determined by:

Algebraic: $\partial_P^i \cdot \partial_P^j \Phi = 0, \quad P_i \cdot \partial_P^j \Phi = 0 \quad i < j, \quad (P_i \cdot \partial_P^i - s_i)\Phi = 0,$

tangent constraint: $X \cdot \partial_P^i \Phi = 0$

radial constraint: $(X \cdot \partial_X + \Delta)\Phi = 0 \qquad \Delta = p + 1 - s$

EOMs and partial gauge: $\Box \Phi = 0, \qquad \partial_P^i \cdot \partial_X \Phi = 0$

Here, p-denotes the height of the uppermost block in the $YT(s_1, \dots, s_{n_1})$, i.e. $s_1 = \dots = s_p > s_{p+1}$.

Gauge symmetries:

$$\delta\Phi = P_\alpha \frac{\partial}{\partial X}\chi^\alpha\,, \qquad \alpha = 1,\dots,p$$

Gauge parameters are constrained. In particular, subject to differential constraints.

For $p > 1$ the gauge symmetry is reducible. Best described in the BRST terms. Ghost variables b_α, gh$(b_\alpha) = -1$. Generating function for fields, parameters and reducibility parameters ("String Field"):

$$\Psi = \Phi(X,P) + b_\alpha\chi^\alpha(X,P) + \sum_{k=2}^{p}\frac{1}{k!}\chi_k^{\alpha_1\dots\alpha_k} b_{\alpha_1}\dots b_{\alpha_k}\,,$$

$\chi_k^{\alpha_1\dots\alpha_k}$- reducibility parameters of order k.

Constraints are unchanged except for:

$$P_i\cdot\partial_P^j \quad\to\quad P_i\cdot\partial_P^j + \delta_i^\alpha\delta_\beta^j b_\alpha\frac{\partial}{\partial b_\beta}\,, \qquad X\cdot\partial_X \quad\to\quad X\cdot\partial_X - b_\alpha\frac{\partial}{\partial b_\alpha}$$

The structure of the gauge algebra is encoded in the BRST operator

$$Q = S^\dagger_\alpha \frac{\partial}{\partial b_\alpha}\,, \qquad Q^2 = 0$$

Well-defined (compatible with constraints on Ψ)

Gauge transformations, reducibility relations are determined by Q, e.g.

$$\delta(\chi^\alpha b_\alpha) = Q(\frac{1}{2}\chi_2^{\alpha\beta} b_\alpha b_\beta) = (P_\alpha \cdot \frac{\partial}{\partial X}\chi_2^{\alpha\beta}) b_\beta$$

Manifestly conformal formulations

For d even $\Delta = 1 + p - s_p = \frac{d}{2} - \ell$ for $\ell = \frac{d-2}{2} + s_p - p$ and the leading b. value is subject to conformal equations $(\Box_0)^\ell \phi_0 + \ldots = 0$. These are generalizations of familiar *Fradkin-Tseytlin* conformal equations *Vasiliev (2009)*.

The same constraints and Q determine boundary conformal gauge system comprising both leading and subleading. In contrast to the scalar case, factorization of the subleading has to be done for fields, gauge parameters and reducibility parameters in a way compatible with Q.

Remarkably, this can be done in terms of $\Psi(X,P,b)$: equivalence relation

$$\Psi \sim \Psi + (X^2)^{\widehat{\ell}}\alpha\,, \qquad \widehat{\ell} = \ell + b_\gamma \frac{\partial}{\partial b_\gamma}$$

and $\alpha(X,P,b)$ satisfies the same constraints except for weight:

$$(X \cdot \partial_X + \frac{d}{2} + \widehat{\ell}\,)\alpha = 0\,.$$

This factorization is compatibles with Q: $Q\bar{\Box}^{\widehat{\ell}}\alpha = \bar{\Box}^{\widehat{\ell}}\beta(\alpha)$ for some $\beta(\alpha)$ satisfying the same constraints as α.

Q defined on the quotient space $\Psi \sim \Psi + (X^2)^{\widehat{\ell}}\,\alpha$ gives a manifestly conformal ($o(d,2)$) formulation of generalized FT gauge fields.

Subtle points:

- Manifestly local description?
  (in AdS_{d+1} or $Conf_d$ terms rather than of $\mathbb{R}^{d,2}\backslash\{0\}$)

- How to avoid partial gauge conditions?

- Too abstract. How to get "usual" form in terms of traceless irreducible Lorentz tensor?

Parent formulation

Idea: the gauge theory is entirely determined by its jet-space stationary surface extended by ghosts and equipped with BRST differential

Vinogradov (1978),...

Vasiliev (1988),...

Barnich, Brandt, Henneaux (1995)

Full scale formalism relevant in the present context:

Barnich, MG, Semikhatov, Tipunin (2004), Barncih, MG (2006)..., MG (2010)

In the present context amount to $\Psi(X,P,b) \to \Psi(X+Y,P,b)$ and $\frac{\partial}{\partial X^A} \to \frac{\partial}{\partial Y^A}$, $X^A \to X^A + Y^A$ and solve all the constraints at fixed X^A e.g. $X^+ = 1, X^- = X^a = 0$. This gives the ghost extended jet-space stationary surface. Q gives BRST differential.

More practically: (using scalar as an illustration)

$$\partial_Y \cdot \partial_Y \Phi = 0\,, \qquad ((X+Y)\cdot \frac{\partial}{\partial Y} + \Delta)\Phi = 0\,, \qquad \nabla\Phi = 0$$

$\nabla = d + \omega^{AB}(X_A + Y_A)\frac{\partial}{\partial Y_B}$.

2nd equation allows to eliminate Y^+, 3rd equation allows to eliminate Y^a, finally the 1st equation gives

Bekaert,MG (2012)

$$\Big(\Box_0 + \frac{\partial}{\partial Y^-}\Big(d - 2\big(\Delta + Y^-\frac{\partial}{\partial Y^-}\big)\Big)\Big)\,\phi(x|Y^-) = 0\,.$$

Nothing but ordinary-derivative formulation of conformal poly-wave equation $\Box_0^\ell = 0$, $\Delta = \frac{d}{2} - \ell$.

Straitforward extension to (partially)-massless fields *Bekaert,MG (2012)*

Analogous to ordinary derivative formulations from *Metsaev (2007),...*

When applied to simplest mixed-symmetry field $YT(2,1)$, $p=1$ and $d=4$ this gives: Then φ_0 satisfies

$$\begin{aligned}\tilde{\Box}^2\varphi_0|_{w=r=0} &= 0,\\ (\partial_p\cdot\partial)\varphi_0 + \frac{\partial}{\partial w}\left(d-\Delta-1+s_1-w\frac{\partial}{\partial w}\right)\varphi_0 + \frac{\partial}{\partial r}(q\cdot\partial_p)\varphi_0 &= 0,\\ (\partial_q\cdot\partial)\varphi_0 + \frac{\partial}{\partial r}\left(d-\Delta-2+s_2-w\frac{\partial}{\partial w}-r\frac{\partial}{\partial r}\right)\varphi_0 &= 0,\\ (\partial_p\cdot\partial_p)\varphi_0 = (\partial_p\cdot\partial_q)\varphi_0 = (\partial_q\cdot\partial_q)\varphi_0 &= 0.\end{aligned}$$

Explicitly in terms of components:

$$\begin{aligned}\Box^2\varphi_{abc} - \Box\partial^e\left(\partial_a\varphi_{ebc}+\partial_b\varphi_{eac}\right) + \frac{1}{2}\Box\partial^e\left(\partial_a\varphi_{bce}+\partial_b\varphi_{ace}\right)\\ -2\Box\partial^e\partial_c\varphi_{abe} + \frac{1}{2}\left(\eta_{ab}\Box+2\partial_a\partial_b\right)\partial^e\partial^f\varphi_{efc}\\ -\frac{1}{4}\partial^e\partial^f\left[\left(\eta_{ac}\Box+2\partial_a\partial_c\right)\varphi_{efb}+\left(\eta_{bc}\Box+2\partial_b\partial_c\right)\varphi_{efa}\right] = 0\end{aligned}$$

The respective Lagrangian was derived using the conventional approach

Alkalaev (2012)

starting from the AdS Lagrangian of *Brink, Metsaev, Vasiliev (2000)*

Conclusions

- We have elaborated on the ambient approach to boundary values and extended it to the case of unitary mixed symmetry AdS fields

- Extension to non-unitary (including partially-massless) mixed symmetry fields. Works in simple cases but general pattern is still missing

- Extension to massive fields, "long" conformal fields *Metsaev (2015)*

- Developing Lagrangian version of the method. More useful in the AdS/CFT context

- Ultimate aim: to be able to move freely between bulk and boundary, especially at the nonlinear level.