Monodromic vs geodesic computation of Virasoro classical conformal blocks

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Outline

- Monodromy approach

Zamolodchikovs' 1994

Fitzpatrick, Kaplan, Walters' 2014

- Geodesic approach

Hijano, Kraus, Snively' 2015

Alkalaev, Belavin' 2015

- Conclusions and outlooks

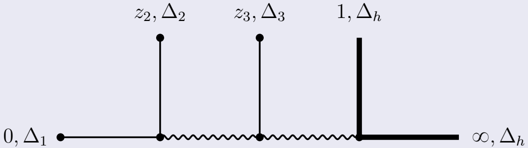
5-point classical Virasoro conformal block

The five-point correlation function of $V_{\Delta_i}(z_i)$, $i = 1, ..., 5$ can be decomposed into conformal blocks

$$\mathcal{F}(z_1, ..., z_5 | \Delta_1, ..., \Delta_5; \tilde{\Delta}_1, \tilde{\Delta}_2; c)$$

which are conveniently depicted as

Fishbone graph



There exist many evidences that in the semiclassical limit $c \to \infty$ the conformal blocks must exponentiate as

$$\mathcal{F}(z_i, \Delta_i, \tilde{\Delta}_j) = \exp\Big[- \frac{c}{6} f(z_i, \epsilon_i, \tilde{\epsilon}_j) \Big] \ ,$$

where $\epsilon_k = \frac{\Delta_k}{c}$ and $\tilde{\epsilon}_k = \frac{\tilde{\Delta}_k}{c}$ are *classical dimensions* and $f(z|\epsilon, \tilde{\epsilon})$ is the *classical* conformal block.

Auxiliary Fuchsian equation

Auxiliary 6-point correlation function $\langle V_{12}(z)V_1(z_1)\cdots V_5(z_5)\rangle$, where $V_{12}(z)$ is the second level degenerate operator. The decoupling condition is

$$\Big[c\frac{\partial^2}{\partial z^2}+\sum_{i=1}^{5}\Big(\frac{\Delta_i}{(z-z_i)^2}+\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\Big)\Big]\langle V_{12}(z)V_1(z_1)\cdots V_5(z_5)\rangle=0\;.$$

In the classical limit $c\to\infty$ the 6-point auxiliary correlation function behaves as

$$\langle V_{12}(z)V_1(z_1)\cdots V_5(z_5)\rangle\Big|_{c\to\infty}\to\psi(z)\exp(-\frac{c}{6}f(z_i,\epsilon,\tilde\epsilon))\;,$$

where $f(z_i)$ is the classical block and $\psi(z)$ is governed by Fuchsian equation

$$\frac{d^2\psi(z)}{dz^2}+T(z)\psi(z)=0\;,\qquad T(z)=\sum_{i=1}^{5}\Big(\frac{\epsilon_i}{(z-z_i)^2}+\frac{c_i}{z-z_i}\Big)\;.$$

Here $T(z)$ is the classical stress-energy tensor and c_i are the accessory parameters

$$c_i(z)=\frac{\partial f(z)}{\partial z_i}\;,\qquad i=1,...,5\;.$$

The asymptotic behaviour $T(z)\sim z^{-4}$ at infinity implies the constraints

$$\sum_{i=1}^{5}c_i=0\;,\qquad\sum_{i=1}^{5}(c_iz_i+\epsilon_i)=0\;,\qquad\sum_{i=1}^{5}(c_iz_i^2+2\epsilon_iz_i)=0\;.$$

Only two accessory parameters are independent, c_2 and c_3.

Heavy-light approximation: Fitzpatrick, Kaplan, Walters' 2014

Let $\epsilon_4 = \epsilon_5 \equiv \epsilon_h$ be the dimension of two heavy fields, while fields with dimensions $\epsilon_1, \epsilon_2, \epsilon_3$ be light. It means that the dimension of heavy operators is fixed in the semiclassical limit while those of light operators tend to zero. The Fuchsian equation can then be solved perturbatively:

$$
\begin{aligned}
\psi(z) &= \psi^{(0)}(z) + \psi^{(1)}(z) + \psi^{(2)}(z) + \ldots \,, \\
T(z) &= T^{(0)}(z) + T^{(1)}(z) + T^{(2)}(z) + \ldots \,, \\
c_i(z) &= c_i^{(0)}(z) + c_i^{(1)}(z) + c_i^{(2)}(z) + \ldots \,,
\end{aligned}
$$

where expansion parameters are light conformal dimensions. In the case of the heavy-light conformal blocks it is sufficient to consider just the first order corrections

$$
\Big(\frac{d^2}{dz^2} + T^{(0)}(z)\Big)\psi^{(0)}(z) = 0 \,, \qquad \Big(\frac{d^2}{dz^2} + T^{(0)}(z)\Big)\psi^{(1)}(z) = -T^{(1)}\psi^{(0)}(z) \,,
$$

where the stress-energy tensor components are directly read off from the main expression. The two branches in the zeroth order are given by

$$
\psi_\pm^{(0)}(z) = (1-z)^{\gamma_\pm} \,, \qquad \gamma_\pm = \frac{1\pm\alpha}{2} \,, \qquad \alpha = \sqrt{1-4\epsilon_h} \,.
$$

Using the method of variation of parameters we find the first order corrections

$$
\begin{aligned}
\psi_+^{(1)}(z) &= \frac{1}{\alpha}\psi_+^{(0)}(z)\int dz\,\psi_-^{(0)}(z)\,T^{(1)}(z)\psi_+^{(0)}(z) - \frac{1}{\alpha}\psi_-^{(0)}(z)\int dz\,\psi_+^{(0)}(z)\,T^{(1)}(z)\psi_+^{(0)}(z) \,, \\
\psi_-^{(1)}(z) &= \frac{1}{\alpha}\psi_+^{(0)}(z)\int dz\,\psi_-^{(0)}(z)\,T^{(1)}(z)\psi_-^{(0)}(z) - \frac{1}{\alpha}\psi_-^{(0)}(z)\int dz\,\psi_+^{(0)}(z)\,T^{(1)}(z)\psi_-^{(0)}(z) \,.
\end{aligned}
$$

Corrections $\psi_\pm^{(1)}(z)$ has branch points identified with punctures at z_2 and z_3.

Contour integration and monodromy

To find the monodromy we evaluate the following integrals

$$I^{(k)}_{++} = \frac{1}{\alpha}\oint_{\gamma_k} dz\, \psi^{(0)}_-(z) T^{(1)}(z) \psi^{(0)}_+(z)\;, \qquad I^{(k)}_{+-} = -\frac{1}{\alpha}\oint_{\gamma_k} dz\, \psi^{(0)}_+(z) T^{(1)}(z) \psi^{(0)}_+(z)\;,$$

$$I^{(k)}_{-+} = \frac{1}{\alpha}\oint_{\gamma_k} dz\, \psi^{(0)}_-(z) T^{(1)}(z) \psi^{(0)}_-(z) \qquad I^{(k)}_{--} = -\frac{1}{\alpha}\oint_{\gamma_k} dz\, \psi^{(0)}_+(z) T^{(1)}(z) \psi^{(0)}_-(z)$$

over two contours γ_2 and γ_3 enclosing points $\{0, z_2\}$ and $\{0, z_2, z_3\}$. For instance, we find

$$I^{(2)}_{+-} = \frac{2\pi i}{\alpha}\left[\alpha\epsilon_1 + c_2(1-z_2) - \epsilon_2 + c_3(1-z_3) - \epsilon_3 - (1-z_2)^{\alpha}[c_2(1-z_2) - \epsilon_2(1+\alpha)]\right]$$

where $c_2 \equiv c^{(1)}_2$ and $c_3 \equiv c^{(1)}_3$. Two monodromy matrices $\mathbb{M} = \{M_{ij},\ i,j = \pm\}$ associated with contours γ_2 and γ_3 are

$$\begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix} \to \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--} \end{pmatrix} \begin{pmatrix} \psi_+(z) \\ \psi_-(z) \end{pmatrix} \qquad \mathbb{M} = \mathbb{M}_0 + \mathbb{M}_1 + \mathbb{M}_2 + \ldots$$

The first order $\mathbb{M}_0$ defines the monodromy of $\psi^{(0)}(z)$. In the linear order the monodromy matrices are given by

$$\mathbb{M}(\gamma_2) = \begin{pmatrix} 1 + I^{(2)}_{++} & I^{(2)}_{+-} \\ I^{(2)}_{-+} & 1 - I^{(2)}_{++} \end{pmatrix}\;, \qquad \mathbb{M}(\gamma_3) = \begin{pmatrix} 1 & I^{(3)}_{+-} \\ I^{(3)}_{-+} & 1 \end{pmatrix}\;.$$

On the other hand, the monodromy matrices over contours γ_2 and γ_3 are defined by the conformal dimensions of the fields in the intermediate channels

$$\tilde{\mathbb{M}}(\gamma_2) = -\begin{pmatrix} e^{+\pi i \Lambda_1} & 0 \\ 0 & e^{-\pi i \Lambda_1} \end{pmatrix} , \qquad \tilde{\mathbb{M}}(\gamma_3) = -\begin{pmatrix} e^{+\pi i \Lambda_2} & 0 \\ 0 & e^{-\pi i \Lambda_2} \end{pmatrix} ,$$

where $\Lambda_1 = \sqrt{1-4\tilde{\epsilon}_1}$ and $\Lambda_2 = \sqrt{1-4\tilde{\epsilon}_2}$ parametrize intermediate dimensions.

Monodromic equations

$$\sqrt{I^{(2)}_{++} I^{(2)}_{++} + I^{(2)}_{+-} I^{(2)}_{-+}} = 2\pi i \, \tilde{\epsilon}_1 , \qquad \sqrt{I^{(3)}_{+-} I^{(3)}_{-+}} = 2\pi i \, \tilde{\epsilon}_2 .$$

Accessory parameters are uniquely defined by 5 algebraic equations which are 3 linear equations and 2 irrational equations.

Superlight expansion

$$c_i = c_i^{(0)} + \epsilon_3 c_i^{(1)} + \epsilon_3^2 c_i^{(2)} + \epsilon_3^3 c_i^{(3)} + \cdots ,$$

where the zeroth-order $c_i^{(0)}$ is the 4-point accessory parameter while $c_i^{(k)}$ are corrections, $k = 1, 2, ...$.

Solving the monodromic equations: $\epsilon_1 = \epsilon_2$ and $\tilde{\epsilon}_1 = \tilde{\epsilon}_2$

Introducing notation

$$x = (1-z_2)c_2\,, \quad y = (1-z_3)c_3 \qquad \text{and} \qquad a = (1-z_2)^\alpha\,, \quad b = (1-z_3)^\alpha$$

and

$$x = \sum_{n=0}^{\infty} \epsilon_3^n x_n\,, \qquad y = \sum_{n=1}^{\infty} \epsilon_3^n y_n\,,$$

we find all corrections up to the third order

$$x_0 = \epsilon_1 + \epsilon_1 \alpha \frac{(a+1)}{(a-1)} + \tilde{\epsilon}_1 \alpha \frac{\sqrt{a}}{a-1}\,, \qquad x_1 = \frac{\alpha}{2}\,\frac{a+b^2}{a-b^2}\,,$$

$$x_2 = \frac{\alpha}{2\tilde{\epsilon}_1}\left[\frac{b\sqrt{a}(a-2ab+b^2)(a-2b+b^2)}{(a-b^2)^3} + \frac{(a-1)(a+b^2)^2}{4\sqrt{a}(a-b^2)^2}\right],$$

$$x_3 = \frac{\alpha}{2\tilde{\epsilon}_1^2}\left[\frac{ab(b-1)(a-2ab+b^2)(a-2b+b^2)(a-3ab+3b^2-b^3)}{(a-b^2)^5}\right],$$

and

$$y_1 = 1 - \alpha\frac{a+b^2}{a-b^2}\,, \qquad y_2 = \frac{\alpha}{\tilde{\epsilon}_1}\left[\frac{b\sqrt{a}(-a+2ab-b^2)(a-2b+b^2)}{(a-b^2)^3}\right],$$

$$y_3 = \frac{\alpha}{2\tilde{\epsilon}_1^2}\left[\frac{b(a-2ab+b^2)(a-2b+b^2)(a^2+a^3-8a^2b+6ab^2+6a^2b^2-8ab^3+b^4+ab^4)}{(a-b^2)^5}\right].$$

Classical conformal block

The power series expansion of the 5-point classical conformal block $f(z)$ is given by

$$f(z) = f^{(0)}(z) + \epsilon_3 f^{(1)}(z) + \epsilon_3^2 f^{(2)}(z) + \epsilon_3^3 f^{(3)}(z) + \ldots \,.$$

Using explicit expressions for the accessory parameters and integrating $c_i = \partial f / \partial z_i$ we find that the expansion coefficients are given by

$$f^{(0)} = -\epsilon_1 \ln \Big[i\, \frac{a-1}{2\sqrt{a}} \Big] + \frac{\epsilon_1}{\alpha} \ln a + \tilde{\epsilon}_1 \ln \Big[i \frac{\sqrt{a}-1}{\sqrt{a}+1} \Big] \,, \quad f^{(1)} = -\ln \Big[-i \frac{a-b^2}{2\sqrt{a} b} \Big] + \frac{1}{\alpha} \ln b \,,$$

$$f^{(2)} = -\frac{1}{\tilde{\epsilon}_1} \frac{(a+b^2)(a+a^2-4ab+b^2+ab^2)}{4\sqrt{a}(a-b^2)^2} \,,$$

$$f^{(3)} = \frac{1}{\tilde{\epsilon}_1^2} \frac{(b-1)b(a-b)(a+b^2)(a+a^2-4ab+b^2+ab^2)}{2(a-b^2)^4} \,,$$

where $a = (1-z_2)^\alpha$ and $b = (1-z_3)^\alpha$. The leading contribution $f^{(0)}$ is the 4-point classical heavy-light conformal block.

The AdS/CFT correspondence

The heavy operators with equal conformal dimensions $\epsilon_n = \epsilon_{n-1} \equiv \epsilon_h$ produce an asymptotically AdS_3 geometry identified either with an angular deficit or BTZ black hole geometry parameterized by

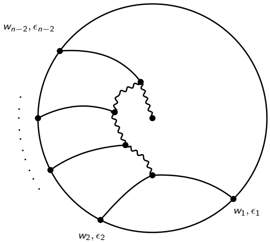
$$\alpha = \sqrt{1 - 4\epsilon_h}$$

The metric reads

$$ds^2 = \frac{\alpha^2}{\cos^2 \rho}\Big(- dt^2 + \sin^2 \rho d\phi^2 + \frac{1}{\alpha^2} d\rho^2 \Big)$$

Here

- $\alpha^2 < 0$ for an angular deficit
- $\alpha^2 > 0$ for the BTZ black hole



The light fields are realized via particular graph of worldlines of $n-3$ classical point probes propagating in the background geometry formed by the two boundary heavy fields. Points w_i are boundary attachments of the light operators.

The identification

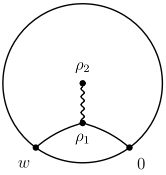
$$S_{cl}^{bulk} \sim f_\delta(z|\epsilon, \tilde{\epsilon}) + \ldots\,, \qquad S_{cl}^{bulk} = \sum_{i=1}^{n-2} \epsilon_i\, L_i + \sum_{i=1}^{n-3} \tilde{\epsilon}_i\, \tilde{L}_i\,,$$

and L_i and $\tilde{L}_i$ are lengths of different geodesic segments on a fixed time slice.

Geodesic approach

The worldline action of a single massive particle with $m \sim \epsilon$ is

$$S = \epsilon \int_{\lambda'}^{\lambda''} d\lambda \sqrt{g_{tt}\dot{t}^2 + g_{\phi\phi}\dot{\phi}^2 + g_{\rho\rho}\dot{\rho}^2}\,, \qquad ds^2 = \frac{\alpha^2}{\cos^2\rho}\Big(-dt^2 + \sin^2\rho d\phi^2 + \frac{1}{\alpha^2} d\rho^2 \Big)$$



Coordinates t and ϕ are cyclic — a constant time disk (ρ,ϕ). Changing variables as $\eta = \cot^2\rho$ and introducing notation $s = \frac{|p_\phi|}{\alpha}$ we find the on-shell action

$$S = \epsilon \ln \frac{\sqrt{\eta}}{\sqrt{1+\eta} + \sqrt{1-s^2\eta}} \Bigg|_{\eta'}^{\eta''}$$

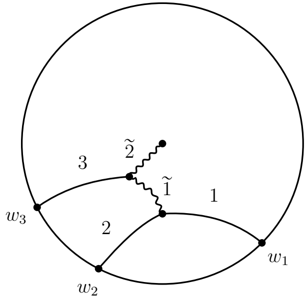
Parameter s is an integration constant that defines a particular form of a geodesic segment.

- The radial line has $s = 0$. For $\rho_1 = \arccos\sin(\alpha w/2)$: $L_{rad} = -\ln\tan\frac{\alpha w}{4}$
- The arc has $s = \cot\frac{\alpha w}{2}$. The length $L_{arc} = \ln\Big[\sin\frac{\alpha w}{2}\Big] + \ln 2\Lambda$
- The 4-pt block: $f \sim \epsilon_{\tilde{1}} L_{rad} + 2\epsilon_1 L_{arc}$

Five-line configuration

The multi-particle action reads

$$S(w) = \epsilon_1 L_1 + \epsilon_2 L_2 + \epsilon_3 L_3 + \epsilon_{\tilde{1}} L_{\tilde{1}} + \epsilon_{\tilde{2}} L_{\tilde{2}}$$



Vertex equilibrium equations

- 1st vertex $\left(\tilde{\epsilon}_1 \tilde{p}^1_\mu + \epsilon_1 p^1_\mu + \epsilon_2 p^2_\mu\right)\Big|_{x=x_1} = 0$
- 2nd vertex $\left(\tilde{\epsilon}_1 \tilde{p}^1_\mu + \tilde{\epsilon}_2 \tilde{p}^2_\mu + \epsilon_3 p^3_\mu\right)\Big|_{x=x_2} = 0$

Angular equations

$$\Delta\phi_1 + \Delta\phi_2 = w_2 - w_1 \;, \quad \Delta\phi_1 + \Delta\phi_3 + \Delta\tilde{\phi}_1 = w_3 - w_1$$

Geodesic equation system

Three linear equations

$$\tilde{s}_2 = 0\,, \qquad \epsilon_3 s_3 - \tilde{\epsilon}_1 \tilde{s}_1 = 0\,, \qquad \epsilon_1 s_1 - \epsilon_2 s_2 - \tilde{\epsilon}_1 \tilde{s}_1 = 0\,,$$

and

Two irrational equations

Vertex eqs

$$\epsilon_3 \sqrt{1 - s_3^2 \eta_2} + \tilde{\epsilon}_1 \sqrt{1 - \tilde{s}_1^2 \eta_2} = \tilde{\epsilon}_2\,, \qquad \epsilon_1 \sqrt{1 - s_1^2 \eta_1} + \epsilon_2 \sqrt{1 - s_2^2 \eta_1} = \tilde{\epsilon}_1 \sqrt{1 - \tilde{s}_1^2 \eta_1}$$

Angular eqs

$$e^{i\alpha w_2} = \frac{\left(\sqrt{1 - s_1^2\, \eta_1} - i s_1 \sqrt{1+\eta_1}\right)\left(\sqrt{1 - s_2^2\, \eta_1} - i s_2 \sqrt{1+\eta_1}\right)}{(1 - i s_1)(1 - i s_2)}$$

$$e^{i\alpha w_3} = \frac{\left(\sqrt{1 - s_3^2 \eta_2} - i s_3 \sqrt{1+\eta_2}\right)\left(\sqrt{1 - \tilde{s}_1^2 \eta_2} - i \tilde{s}_1 \sqrt{1+\eta_2}\right)\left(\sqrt{1 - s_1^2 \eta_1} - i s_1 \sqrt{1+\eta_1}\right)}{(1 - i s_3)\left(\sqrt{1 - \tilde{s}_1^2 \eta_1} - i \tilde{s}_1 \sqrt{1+\eta_1}\right)(1 - i s_1)}$$

- 5-pt case: a complicated higher order algebraic equation
- 4-pt case: an exact solution (Hijano, Kraus, Snively, 2015)

Monodromy vs geodesic approach

Computing the geodesic length *vs* integrating canonical momenta in the attachment points.

There are three boundary attachments $w_1 = 0$ and w_2, w_3 so that

$$\alpha\epsilon_2 s_2(w_2, w_3) = \frac{\partial S(w_2, w_3)}{\partial w_2}\;, \qquad \alpha\epsilon_3 s_3(w_2, w_3) = \frac{\partial S(w_2, w_3)}{\partial w_3}$$

The accessory parameters are defined in much the same way as

$$c_2(z_2, z_3) = \frac{\partial f(z_2, z_3)}{\partial z_2}\;, \qquad c_3(z_2, z_3) = \frac{\partial f(z_2, z_3)}{\partial z_3}$$

The two systems above define potential vector fields in two dimensions which can be related to each other.

- Coordinates

$$w_m = i \ln(1 - z_m)\;, \qquad m = 1, 2, 3$$

- Potentials

$$f(z_2, z_3) = S(w_2, w_3) + i\epsilon_2 w_2 + i\epsilon_3 w_3$$

It follows that the accessory and angular momenta parameters are related as

$$c_m = \epsilon_m \frac{1 \pm i\alpha s_m(w)}{1 - z_m}\;, \qquad m = 1, 2, 3$$

The above map can be considered as an AdS/CFT correspondence.

The differential equations are easy to integrate while parameters satisfy complicated equations.

A physical root

- Within the monodromy approach there are five variables $c_1, ..., c_5$ (accessory parameters) subjected to three linear and two irrational equations

$$M_\alpha(c) = 0\;, \qquad \alpha = 1, ..., 5$$

- Within the geodesic approach there are seven variables $s_1, s_2, s_3, \tilde{s}_1, \tilde{s}_2$ (external/intermediate angular momenta) and η_1, η_2 (radial vertex positions) subjected to three linear and four irrational equations

$$G_I(s, \tilde{s}, \eta) = 0 \qquad I = 1, ..., 7$$

In principle, one might expect that eliminating the vertex position variables the residual two geodesic equations match exactly with the monodromic equations. Instead, a weaker version of the equivalence turns out to be true – the systems are required to have at least one common root. It is instructive to have both monodormic and geodesic equations expressed in the same notation.

The 4-point case

Monodromic equation: $$\Big(s + i\frac{(a+1) - \sqrt{a}\varkappa}{1-a}\Big)^2 = 0\;,$$

Geodesic equation: $$(s+i)(s + i\frac{(a+1) - \sqrt{a}\varkappa}{1-a}) = 0\;,$$

where $a = (1-z_2)^\alpha$. The above equations do not coincide but have a common root.

The 5-point case

By analogy with the monodromic equations the geodesic ones have no explicit solution.

- All linear geodesic equations are explicitly mapped to linear monodromic equations.
- A combination of geodesic irrational equations have a root which is exactly mapped to one irrational monodoromic equation.
- The rest of geodesic irrational equations allows just for a perturbative analysis.

The expansion of angular momenta up to the third order is given by

$$s_i = s_i^{(0)} + \nu s_i^{(1)} + \nu^2 s_i^{(2)} + \nu^3 s_i^{(3)} + \ldots \,, \qquad \nu = \epsilon_3/\tilde{\epsilon}_1 \,, \qquad i = 2,3$$

The expansion coefficients are found to be (here $\varkappa = \tilde{\epsilon}_1/\epsilon_1$ and $\theta_{2,3} = \alpha w_{2,3}/2$)

$$s_2^{(0)} = -\cot\theta_2 + \varkappa \frac{1}{2\sin\theta_2} \,, \qquad s_2^{(1)} = \frac{\varkappa}{2}\cot(2\theta_3 - \theta_2) \,,$$

$$s_2^{(2)} = \varkappa \frac{[9\cos(2\theta_3) + 7\cos(2\theta_2 - 2\theta_3) - \cos(4\theta_2 - 6\theta_3) + \cos(2\theta_2 - 6\theta_3) - 4\cos(2\theta_2 - 4\theta_3) - 12]}{32\sin^3(\theta_2 - 2\theta_3)}$$

$$s_2^{(3)} = \varkappa \frac{\sin\theta_3[\sin(\theta_2 - 3\theta_3) - 3\sin(\theta_2 - \theta_3)][3 + \cos(2\theta_2 - 4\theta_3) - 2\cos(2\theta_2 - 2\theta_3) - 2\cos(2\theta_3)]}{8\sin^5(\theta_2 - 2\theta_3)} \,,$$

$$s_3^{(0)} = -\cot(2\theta_3 - \theta_2) \,, \qquad s_3^{(1)} = \frac{1}{2}\csc^3(\theta_2 - 2\theta_3)[\sin^2\theta_2 + 4\sin^2(\theta_2 - \theta_3)\sin^2\theta_3] \,,$$

$$s_3^{(2)} = -\frac{1}{16}\csc^5(\theta_2 - 2\theta_3))[6\cos\theta_2 + \cos(\theta_2 - 4\theta_3) + \cos(3\theta_2 - 4\theta_3) - 8\cos(\theta_2 - 2\theta_3)]\times$$

$$\times[3 + \cos(2\theta_2 - 4\theta_3) - 2\cos(2\theta_2 - 2\theta_3) - 2\cos(2\theta_3)] \,.$$

Multi-particle action

The power series expansion of the bulk multi-particle action $S(w)$ is given by

$$S(w) = S^{(0)}(w) + \epsilon_3 S^{(1)}(w) + \epsilon_3^2 S^{(2)}(w) + \epsilon_3^3 S^{(3)}(w) + \dots \, .$$

Using explicit expressions for the angular momenta and integrating $\alpha \epsilon_i s_i = \partial S / \partial w_i$ we find the expansion coefficients are given by

$$S_0(\theta) = -2\epsilon_1 \ln \sin \theta_2 + \tilde{\epsilon}_1 \ln \tan \frac{\theta_2}{2} \, , \qquad S_1(\theta) = -\ln \sin(2\theta_3 - \theta_2) \, ,$$

$$S_2(\theta) = -\frac{\cos \theta_2 + 2 \csc^2(\theta_2 - 2\theta_3) \sin(\theta_2 - \theta_3) \sin \theta_3}{2\tilde{\epsilon}_1} \, ,$$

$$S_3(\theta) = -\frac{\cos \theta_2 + 2 \csc^2(\theta_2 - 2\theta_3) \sin(\theta_2 - \theta_3) \sin \theta_3}{2\tilde{\epsilon}_1} \times \frac{4 \csc^2(\theta_2 - 2\theta_3) \sin(\theta_2 - \theta_3) \sin \theta_3}{2\tilde{\epsilon}_1} \, ,$$

where we switched to $\theta_{2,3} = \alpha w_{2,3}/2$.

- The above expansion coefficients are related to the conformal block according to the general identification formula.
- NB! The same results follow from the explicit geodesic length formula.

Conclusions & outlooks

Conclusions

- We have computed the 5-point heavy-light conformal block in the super-light approximation up to the third order with respect to the conformal dimension of one of the three light fields. The computation has been done in two independent ways: using the monodromy and the geodesic approaches. The resulting expressions coincide.
- We observe different aspects of the correspondence between the two methods. In particular, we find that the boundary variables and equations have their counterparts in the bulk consideration. There is also a precise relation between the accessory parameters and the conserved angular momenta of the different geodesic segments.

Outlooks

The similarity between bulk and boundary computations leads to the natural assumption that in the present context the AdS_3/CFT_2 correspondence is to be understood in a strong sense, *i.e.* as two different descriptions of the same Liouville theory in the semiclassical limit $c \to \infty$.