# Star-Product Functions in HS Theory and Locality


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# Higher Derivatives in HS interactions

HS interactions contain higher derivatives

Nonanaliticity in $\Lambda$ via dimensionless combination $\Lambda^{-\frac{1}{2}}\frac{\partial}{\partial x}$

By a seemingly local field redefinition (Prokushkin, MV 1998) induced by the integrating flow it is possible to get rid of currents from HS field equations including the stress tensor in the spin-two sector: the field transformation induced by the integrating flow is nonlocal having the form

$$\phi \to \phi' = \phi + \sum_n a_{nm}(\rho D)^n \phi \, (\rho D)^m \phi + \dots \,,$$

$\rho$ is the $AdS$ radius, $D$ is the space-time covariant derivative.

The problem: find restrictions on $a_{nm}$ distinguishing between truly non-local and generalized local field redefinitions containing an infinite number of terms but $a_{nm}$ decrease fast enough with $n$ and $m$.

The problems in $AdS_d$ and Minkowski space are essentially different

# Greens function example

Consider a massive field equation

$$(\Box + m^2)\phi = 0$$

The Green function

$$G = (\Box + m^2)^{-1}$$

can be represented in the pseudolocal form

$$G = m^{-2} \sum_{n=0}^{\infty} \left(-\frac{\Box}{m^2}\right)^n$$

Green function is non-local: not decreasing expansion coefficients imply nonlocality.

$m^2$ is a counterpart of $\Lambda$ for massless particles in $AdS$

The idea: to look for a class of field redefinitions which

- are closed under successive application: form an algebra

- rule out obviously nonlocal field redefinitions like integration flow

In the unfolded form of HS theories the space-time dependence is encoded in twistor-like variables $Z^A$ and $Y^A$. The problem is to find restrictions on the coefficients $b_{nmkl}$ in

$$\phi \to \phi' = \phi + \sum_{nmkl} b_{nmkl} \Big( \Big(\frac{\partial}{\partial Z}\Big)^n \Big(\frac{\partial}{\partial Y}\Big)^m \phi \Big) \Big( \Big(\frac{\partial}{\partial Z}\Big)^k \Big(\frac{\partial}{\partial Y}\Big)^l \phi \Big) + \dots$$

# Nonlinear HS equations

$$\mathcal{W}(Z;Y;k,\bar{k}|x) = (\mathsf{d}+W)+S\,, \qquad W = dx^n W_n\,, \qquad S = \theta^\alpha S_\alpha + \bar{\theta}^{\dot\alpha}\bar{S}_{\dot\alpha}$$

$$\mathcal{W}\star\mathcal{W} = i(\theta^A\theta_A + \eta\theta^\alpha\theta_\alpha B\star k\star\kappa + \bar{\eta}\bar{\theta}^{\dot\alpha}\bar{\theta}_{\dot\alpha}B\star\bar{k}\star\bar{\kappa})$$

$$\mathcal{W}\star B = B\star\mathcal{W}\,, \qquad B = B(Z;Y;k,\bar{k}|x)$$

**HS star product**

$$(f\star g)(Z;Y) = \frac{1}{(2\pi)^4}\int d^4U\,d^4V\,\exp\,[iU_AV^A]\,f(Z+U;Y+U)g(Z-V;Y+V)$$

**This is the normal-ordered product with respect to** $Y\pm Z$

# Klein operators and supertrace

## Klein operator

$$\kappa = \exp i z_\alpha y^\alpha\,, \qquad \kappa \star \kappa = 1$$

$$\kappa \star f(z,y) = f(-z,-y) \star \kappa$$

## Supertrace

$$str(f(z,y)) = \frac{1}{(2\pi)^2} \int d^2u\, d^2v \exp\left[-iu_\alpha v^\beta\right] f(u,v)$$

$$str(f \star g) = str(g \star f)$$

## Klein operators have well-defined star product but divergent supertrace

$$str(\kappa) \sim \delta^4(0)$$

# Perturbative analysis

The standard vacuum solution is $B = 0$ and

$$\mathcal{W}_0 = \mathsf{d}_x + Q + W_0(Y|x)\,, \qquad Q := \theta^A Z_A$$

The space-time one-form $W_0(Y|x)$ solves the flatness equation

$$\mathsf{d}_x W_0(Y|x) + W_0(Y|x) \star W_0(Y|x) = 0\,.$$

The star-commutator with $Q$ yields de Rham derivative in $Z^A$

$$Q \star f(Z;Y) - (-1)^{deg_f} f(Z;Y) \star Q = -2i\mathsf{d}_Z f(Z;Y)\,, \qquad \mathsf{d}_Z = \theta^A \frac{\partial}{\partial Z^A}$$

Standard homotopy formula:

$$\mathsf{d}_Z g(\theta_Z; Z; Y) = f(\theta_Z; Z; Y) \quad \Longrightarrow \quad g(\theta_Z; Z; Y) = \partial^*_Z f + \mathsf{d}_Z \varepsilon + g(0;0;Y)\,,$$

$$\partial^*_Z f := \mathsf{d}^*_Z H(f)\,, \qquad H(f) := \int_0^1 d\tau \tau^{-1} f(\tau\theta_Z; \tau Z; Y)\,, \qquad \mathsf{d}^*_Z = Z^A \frac{\partial}{\partial \theta^A}\,.$$

$\mathsf{d}_Z \varepsilon$: exact forms

$g(0;0;Y)$: de Rham cohomology

**Via homotopy formula Klein operators generate perturbative solution**

$$f(Z;Y)=\int_0^1 d\tau\varphi(Z;Y;\tau)\exp i\tau Z_A Y^A\,,$$

$$\varphi(Z;Y;\tau)=\sum_{n,m=0}^{\infty}\varphi_{A_1,\dots,A_n,B_1,\dots,B_m}(\tau)Z^{A_1}\dots Z^{A_n}Y^{B_1}\dots Y^{B_m}$$

**with the coefficients $\varphi_{A_1,\dots,A_n,B_1,\dots,B_m}(\tau)$ integrable in $\tau$. Distributions in $\tau$ are allowed. Behavior of $\varphi_{A_1,\dots,A_n,B_1,\dots,B_m}(\tau)$ determine properties of $f(Z;Y)$ with respect to twistorial variables $Z,Y$ and, by virtue of the unfolded equations, in space-time coordinates $x$.**

$$(f_1\star f_2)(Z;Y)=\int d\tau_{1,2}\varphi_{1,2}(Z;Y;\tau_{1,2})\exp i\tau_{1,2}Z_A Y^A\,,$$

$$\varphi_{1,2}(Z;Y;\tau_{1,2})=\frac{1}{(2\pi)^M}\int d\tau_1 d\tau_2 dS dT\delta(\tau_{1,2}-\tau_1\diamond\tau_2)\exp iS_A T^A$$
$$\varphi_1((1-\tau_2)Z-\tau_2 Y+S;(1-\tau_2)Y-\tau_2 Z+S;\tau_1)$$
$$\varphi_2((1-\tau_1)Z+\tau_1 Y-T;\tau_1 Z+(1-\tau_1)Y+T;\tau_2)\,,$$

$$a\diamond b=a+b-2ab=a(1-b)+b(1-a)$$

**is commutative $a\diamond b=b\diamond a$ and associative product in** $\mathbb{R}$ or $\mathbb{C}$

# Key observation

The idea is to specify appropriate classes of functions $\varphi(Z;Y;\tau)$.

The key observation is that the space $V_{0,0}$ of functions of the form

$$f(Z;Y) = \int_0^1 d\tau \phi(\tau Z; (1-\tau)Y;\tau) \exp i\tau Z_A Y^A$$

with $\phi(W;U;\tau)$ regular in $W$ and $U$ and integrable in $\tau$ is closed under the HS star product. Being accompanied by the factor of $\tau$ and $1-\tau$ the dependence on $Z$ and $Y$ trivializes at $\tau \to 0$ and $\tau \to 1$,

Such behavior is appropriate for the perturbative analysis of HS theory.

Coefficients of $Z^n Y^m$ will contain the decreasing factor

$$\int_0^1 dt t^n (1-t)^m = \beta(n,m) = \frac{n!m!}{(n+m+1)!}$$

# Proof

**For** $f_{1,2} \in V_{0,0}$

$$\varphi_{1,2}(W;U;\tau_{1,2}) = \frac{1}{(2\pi)^M} \int dS dT \exp iS_A T^A \int_0^1 d\tau_1 d\tau_2 \delta(\tau_{1,2} - \tau_1 \diamond \tau_2)$$

$$\phi_1(\tau_1[(1-\tau_2)W - \tau_2 U + S]; (1-\tau_1)[(1-\tau_2)U - \tau_2 W + S]; \tau_1)$$

$$\phi_2(\tau_2[(1-\tau_1)W + \tau_1 U - T]; (1-\tau_2)[\tau_1 W + (1-\tau_1)U + T]; \tau_2)$$

**Elementary relations**

$$\tau_1 \diamond \tau_2 = (1-\tau_1)\tau_2 + (1-\tau_2)\tau_1 \,,$$

$$1 - \tau_1 \diamond \tau_2 = (1-\tau_1)(1-\tau_2) + \tau_1\tau_2$$

**imply the square decomposition inequalities**

$$(1-\tau_1)\tau_2 = \alpha(\tau_1,\tau_2)\tau_1 \diamond \tau_2 \,, \qquad (1-\tau_2)\tau_1 = \beta(\tau_1,\tau_2)\tau_1 \diamond \tau_2 \,,$$

$$(1-\tau_1)(1-\tau_2) = \gamma(\tau_1,\tau_2)(1-\tau_1 \diamond \tau_2) \,, \qquad \tau_1\tau_2 = \rho(\tau_1,\tau_2)(1-\tau_1 \diamond \tau_2)$$

$$\alpha(\tau_1,\tau_2),\ \beta(\tau_1,\tau_2),\ \gamma(\tau_1,\tau_2),\ \rho(\tau_1,\tau_2) \in [0,1] \,.$$

**Hence** $f_1, f_2 \in V_{0,0}$: $f_1 * f_2 \in V_{0,0}$

# Spaces $V_{k,l}$

$V_{k,l}$ **is the space of such star-product elements that**

$$\phi(W;U;\tau) = \tau^k (1-\tau)^l \phi'(W;U;\tau)$$

**A (poly)logarithmic dependence on $\tau$ and $1-\tau$ does not affect $k$ and $l$**

**The spaces $V_{k,l}$ have the fundamental composition property**

$$V_{k_1,l_1} \star V_{k_2,l_2} \subset V_{\mathrm{min}(k_1,l_2)+\mathrm{min}(k_2,l_1)+1\,,\,\mathrm{min}(k_1,k_2)+\mathrm{min}(l_1,l_2)+1}$$

**This follows from square decomposition inequalities along with the fact that the integral**

$$\int_0^1 d\tau_1 \int_0^1 d\tau_2 \delta(\tau - \tau_1 \diamond \tau_2) = -\log((1-2\tau)^2)$$

**behaves as $\tau$ at $\tau \to 0$ and $1-\tau$ at $\tau \to 1$**

# Spaces $V_{k,l,p}$

For $\theta$-dependent differential forms $f(\theta_Z;Z;Y) \in V_{k,\,l,p}$ if $f(\theta_Z;Z;Y)$ is a $p$-form with coefficients in $V_{k,\,l}$.

$$Q \in V_{-2,\,\infty,1}$$

because both $\delta(\tau)$ and $\tau^{-1}$ in $Q = \int_0^1 d\tau\delta(\tau)\tau^{-1}(\tau\theta^A Z_A)$ bring negative contribution to the first index of $V_{-2,\,l,1}$. Since $[Q\,,\ldots]_* \sim \mathsf{d}_Z = \theta^A \frac{\partial}{\partial Z^A}$,

$$[Q\,,V_{k,l,p}]_\star \subset V_{k+1,l-1,p+1}\,.$$

For $Q$-closed $f \in V_{k,l,p}$ a solution to $\mathsf{d}_Z g = f$ is

$$\partial^*_Z f(\theta_Z;Z;Y) = Z^A \frac{\partial}{\partial\theta^A}\int_0^1 \frac{ds}{s}\int_0^1 d\tau\phi(s\theta_Z;s\tau Z;(1-\tau)Y;\tau)\exp is\tau Z_A Y^A$$

An elementary analysis shows

$$\partial^*_Z V_{k,l,p} \subset V_{min(p-1,k)-1\,,\,l+1\,,\,p-1}$$

# Field algebra

**HS field algebra $\mathcal{H}$ appropriate for perturbative analysis of HS equations:**

$$\mathcal{H} := \oplus_{p=0}^{M}\mathcal{H}_p\,, \qquad \mathcal{H}_p := V_{p-1,M-p-1,p}\,.$$

**Using that any $p$-form in $\theta_Z$ with $p > M$ is zero it follows**

$$\mathcal{H}_p \star \mathcal{H}_q \subset \mathcal{H}_{p+q} \qquad \Longrightarrow \qquad \mathcal{H}\star\mathcal{H}\subset\mathcal{H}$$

$$[Q\,,\mathcal{H}_p]_\star \subset \mathcal{H}_{p+1}\,, \qquad \partial_Z^*\mathcal{H}_p \subset \mathcal{H}_{p-1}\,, \qquad \partial_Z^*\mathcal{H}_0 = 0$$

**Although $\mathcal{H}$ is invariant under the action of the homotopy operator $\partial_Z^*$ and derivative $\mathsf{d}_Z$, $Q \in V_{-2,\forall l,1} \notin \mathcal{H}$ $Q$ induces outer derivation of $\mathcal{H}$. The HS connection $\mathcal{W}$ should be written in the form**

$$\mathcal{W} = \mathsf{d}_x + Q + \mathcal{W}'\,, \qquad \mathcal{W}' \in \mathcal{H}\,.$$

*Theorem* : **Since $\theta^\alpha\theta_\alpha \star \kappa \in \mathcal{H}$, $\bar\theta^{\dot\alpha}\bar\theta_{\dot\alpha}\star\bar\kappa \in \mathcal{H}$ $\mathcal{W}'$ and $\mathcal{B}$ resulting from the perturbative solution of the HS equations belong to $\mathcal{H}$**

# Local HS algebra

$$\mathcal{H}^{loc} = \oplus_{p=0}^{M} \mathcal{H}_p^{loc}\,, \qquad \mathcal{H}_p^{loc} \subset V_{p-1,M-p-1+\epsilon,p}\,, \qquad \forall \epsilon > 0$$

**The difference between $\mathcal{H}^{loc}$ and $\mathcal{H}$ is dominated by any rational behavior in $1-\tau$.**

$$\mathcal{H}_p^{loc} \star \mathcal{H}_q^{loc} \subset \mathcal{H}_{p+q}^{loc}$$

*Theorem* : **$\mathcal{H}^{loc}$ is an algebra invariant under the action of the homotopy operator $\partial_Z^*$ and $\mathsf{d}_Z$.**

**It follows that the supertrace of elements of $\mathcal{H}_p$ diverges as**

$$str(\mathcal{H}_p) \sim \int_0^1 d\tau (1-\tau)^{-1-p} \ldots .$$

*Theorem* : **Elements of $\mathcal{H}_0^{loc}$ have well-defined supertrace.**

# Locality conjecture

A field redefinition $\phi \to \phi' = f(\phi)$

$$f(\phi) = f + \sum_{g,h,\dots} \left(g_1 \star \phi \star g_2 + h_1 \star \phi \star h_2 \star \phi \star h_3 + \dots\right),$$

of $\phi = \mathcal{W}', B \in \mathcal{H}$ where summation is over various $f, g, h, \dots$. $f(\phi)$ is local if $f, g, h, \dots \in \mathcal{H}^{loc}$, minimally nonlocal if $f, g, h, \dots \in \mathcal{H}$ and strongly nonlocal otherwise. Since $(\mathcal{H})\mathcal{H}^{loc}$ is an algebra, the composition of any two local transformations is local.

Conjecture I: *local* transformations provide a proper generalization of the local transformations in Minkowski space.

The integrating flow of Prokushkin, MV (1998) is nonlocal

Conjecture II: gauge transformations with parameters in $\mathcal{H}$ are allowed.

Those beyond $\mathcal{H}$ are not: specification of allowed gauge transformation is important for application of quasi gauge transformations that “gauge away” the space-time dependence: the final answer should belong to $\mathcal{H}$.

# HS star product versus Weyl

## Formal map to the Weyl star product

$$f_W(Z;Y) = \frac{1}{(2\pi)^M}\int dS dT \,\mathrm{exp}\, -iS_A T^A f_{HS}(Z+S;Y+T)\,,$$

**Being equivalent for polynomials, different star products may be inequivalent beyond this class.**

## Weyl-Moyal star product

$$(f_W \star g_W)(Z;Y) = \frac{1}{(2\pi)^{2M}}\int dU dV \,\mathrm{exp}\,[i(-U_{1A}V_1^A + U_{2A}V_2^A)]$$
$$f_W(Z+U_1;Y+U_2)g_W(Z+V_1;Y+V_2)$$

## The map is singular at $Z \neq 0$

$$f_W(Z;Y) = \frac{1}{(2\pi)^M}\int_0^1 d\tau(1-\tau)^{-M}\int dS dT \,\mathrm{exp}\,[-iS_A T^A + i\frac{\tau}{1-\tau}Z_A Y^A]$$
$$\phi\Big(\tau S + \frac{\tau}{1-\tau}Z;Y+T;\tau\Big)$$

# Conclusion

Classes of star-product functions are identified distinguishing between local and nonlocal field redefinitions in HS theory

The r.h.s.s of HS field equations are in the De Rham cohomology with respect to the local class