On Partition Functions of Higher Spin Theories

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with M. Beccaria

Motivation:
learn about
(i) quantum HS theories
(ii) limits of AdS/CFT

Free Higher Spin theory

Flat space background:

consider collection of free massless spin $s = 0, 1, 2, ...$ fields

with gauge-invariant $\delta\varphi_{m_1...m_s} = \partial_{(m_1}\epsilon_{m_2...m_s)}$

Fronsdal action $S = \int d^4x\, \partial^n\varphi^{m_1...m_s}\partial_n\varphi_{m_1...m_s} + ...$

e.g. viewed as formal flat limit of Vasiliev HS theory with no interactions:

massless vector, massless graviton, etc.: for $s > 0$ 2 d.o.f. in d $= 4$

curious fact: total number of d.o.f. is zero

$$1 + \sum_{s=1}^{\infty} 2 = 1 + 2\zeta_R(0) = 0$$

free massless spin s partition function

$$Z_{\mathrm{MHS},s} = \Big[\frac{\det\Delta_{s-1\,\perp}}{\det\Delta_{s\,\perp}}\Big]^{1/2} = \Big[\frac{(\det\Delta_{s-1})^2}{\det\Delta_s \det\Delta_{s-2}}\Big]^{1/2} = \Big([\det(-\partial^2)]^{-1/2}\Big)^2$$

$\Delta_s = -\partial^2$ on symmetric rank s traceless tensor e.g. Maxwell vector:

$$L = \frac{1}{4} F_{mn} F^{mn} = \frac{1}{2} A^{\perp}_{m} (-\partial^2) A^{\perp}_{m}$$

$$Z = \int [dA] \exp[-\tfrac{1}{2} \int d^4x \ A^{\perp}_{m} (-\partial^2) A^{\perp}_{m}]$$

$$A_m = A^{\perp}_{m} + \partial_m \phi, \qquad A^2 = A^{\perp} A^{\perp} + \phi(-\partial^2)\phi$$

$$dA = dA^{\perp} d\phi \, [\det(-\partial^2)]^{1/2} , \qquad \int dA \ e^{-A^2} = 1$$

$$[dA] = dA^{\perp} = \frac{dA}{d\phi}$$

$$Z = \Big[\frac{\det \Delta_0}{\det \Delta_{1\,\perp}} \Big]^{1/2}$$

$$\Delta_0 = -\partial^2 , \qquad \Delta_{1\,\perp} = P^{\perp} \Delta_0$$

Then total partition function is trivial:

$$(Z_{\rm MHS})_{\rm tot} = \prod_{s=0}^{\infty} Z_{{\rm MHS},s}$$
$$= \Big[\frac{1}{\det \Delta_0}\Big]^{1/2}\Big[\frac{\det \Delta_0}{\det \Delta_{1\,\perp}}\Big]^{1/2}\Big[\frac{\det \Delta_{1\,\perp}}{\det \Delta_{2\,\perp}}\Big]^{1/2}\Big[\frac{\det \Delta_{2\,\perp}}{\det \Delta_{3\,\perp}}\Big]^{1/2} ... = 1$$

• cf. supersymmetric theory: B/F =1 (e.g. vanishing of vacuum energy)

• here cancellation of physical spin s det and ghost det for spin $s+1$ field

• should be reflecting large gauge symmetry of the theory
(cf. topological theory like antisymm tensor of rank d in $d+1$ dimensions
or Chern-Simons or 3d gravity)

• cancellation of an infinite number of factors is formal (like 1-1+1-1+...=0):
depends on grouping terms together – ∞ product requires regularization
and its value may depend on choice

• choice of regularization should be consistent with underlying symmetry:
here with higher spin gauge symmetry

case of $\mathrm{d}=4$:

$$Z_0 = \Big[\frac{1}{\det \Delta_0}\Big]^{1/2}, \qquad Z_{\mathrm{MHS},s} = (Z_0)^{\nu_s}\ , \qquad \nu_s = 2$$

$$\nu_s = (s+1)^2 + (s-1)^2 - 2s^2 = 2$$

$$Z_{\rm tot} = (Z_0)^{\nu_{\rm tot}}\ , \qquad \nu_{\rm tot} = 1 + \sum_{s=1}^{\infty} \nu_s = 1 + \sum_{s=1}^{\infty} 2 = 0$$

• $\mathrm{d}=4$: ζ-function reg. is equivalent to formal cancellation of factors in Z
• cf. use of ζ-function regularization in vac energy in bosonic string:
consistent with massless vector in $d=26$ – symmetries of critical string
• in d flat dimensions:

$$\det \Delta_s = (\det \Delta_0)^{N_s}, \ \ \det \Delta_{\perp\, s} = (\det \Delta_0)^{N_s^\perp}\ , \ \ N_s = \binom{s+\mathrm{d}-1}{s} - \binom{s+\mathrm{d}-3}{s-2}$$

$$N_s^\perp = N_s - N_{s-1}, \quad \nu_s = N_s^\perp - N_{s-1}^\perp = 2[s + \tfrac{1}{2}(\mathrm{d}-4)]\,\frac{(s+\mathrm{d}-5)!}{s!(\mathrm{d}-4)!}$$

in even d one may use regularization ($\epsilon \to 0$, dropping singular terms)

$$\nu_{\rm tot} = 1 + \sum_{s=1}^{\infty} \nu_s \, e^{-\epsilon[s+\frac{1}{2}({\rm d}-4)]}\Big|_{\rm fin.} = 0$$

• alternative reg. in any d: cutoff function $f(s,\epsilon)$ with $f(s,0) = 1$ for $\Delta_{\perp,s}$
$\nu_{\rm tot} = 1 + \sum_{s=1}^{\infty} \left[f(s,\epsilon) \, N_s^{\perp} - f(s-1,\epsilon) \, N_{s-1}^{\perp} \right] = 0$
direct analog of formal cancellation of the determinant factors $Z_{\rm tot}$

Regularization vs symmetry:

• ${\rm d} = 4 + n$ dimensional theory: regularization should preserve d-dimensional Lorentz symmetry

• if viewed as 4d theory + KK modes $\{\phi_n\}$ requires special regularization of sum over KK mode number n (cf. no log divergences in 5d vs 4d)

• analogy between higher spins $\{\varphi_s\}$ and KK modes [Fronsdal]:
HS symmetry requires special regularization of sum over s
or doing computation without splitting into 4d fields with fixed s

Conformally-flat case: $AdS_{\rm d}$

$Z_{\rm tot} = 1$ holds also in proper vacuum of Vasiliev theory – $AdS_{\rm d}$

Fronsdal action in $AdS_{\rm d}$ leads to similar partition function

basic kinetic operator in AdS_d ($k = 0, 1, ..., s-1$)

$$\Delta_s(M^2_{s,k}) \equiv -\nabla^2_s + M^2_{s,k}\varepsilon \qquad\qquad M^2_{s,k} = s - (k-1)(k+{\rm d}-2)$$

$\varepsilon = \pm 1$ for unit-radius $S^{\rm d}$ or euclidean $AdS_{\rm d}$; $\varepsilon = 0$ in flat space

Partition function of "partially-massless" field (rank k gauge parameter)

$$Z_{s,k} = \Big[\frac{\det \Delta_{k\,\perp}(M^2_{k,s})}{\det \Delta_{s\,\perp}(M^2_{s,k})}\Big]^{1/2}$$

Massless (maximal gauge invariance with rank $s-1$ parameter)

spin s field on homogeneous conformally flat space

[Gaberdiel et al 2010; Gupta, Lal 2012; Metsaev 2014]

$$Z_{{\rm MHS},s} = Z_{s,s-1} = \Big[\frac{\det \Delta_{s-1\,\perp}(M^2_{s-1,s})}{\det \Delta_{s\,\perp}(M^2_{s,s-1})}\Big]^{1/2}$$

$$Z_{\mathrm{MHS},s} = \Big[\frac{\big(\det \Delta_{s-1}(M^2_{s-1,s})\big)^2}{\det \Delta_s(M^2_{s,s-1}) \det \Delta_{s-2}(M^2_{s+2,s+1})}\Big]^{1/2}$$

$$Z_{\mathrm{MHS},0} = [\det{(-\nabla^2 + M_0^2)}]^{-1/2}, \qquad M_0^2 = 2(\mathrm{d}-3)\varepsilon$$

$$Z_{\mathrm{tot}} = \prod_{s=0}^{\infty} Z_{\mathrm{MHS},s}$$

here no immediate cancellation of factors: different for $\varepsilon \neq 0$

Using spectral ζ-function (Λ is UV cutoff, r is curvature radius)

$$\ln\det \Delta_s = -\zeta_{\Delta_s}(0)\ln(\Lambda^2 r^2) - \zeta'_{\Delta_s}(0)$$

Computing $\zeta_{\mathrm{tot}}(z) = \sum_{s=0}^{\infty} \zeta_{\Delta_s}(z)$ and then taking $z \to 0$:

$\zeta_{\mathrm{tot}}(z) = 0 + 0 \times z + \mathcal{O}(z^2)$ [Giombi, Klebanov, Safdi: 2014]

$$\Big(Z_{\mathrm{MHS}}(AdS_{\mathrm{d}})\Big)_{\mathrm{tot}} = 1$$

Equivalent regularization:

$$\ln\Big(Z_{\mathrm{MHS}}(AdS_{\mathrm{d}})\Big)_{\mathrm{tot}} = \sum_{s=0}^{\infty} \ln Z_{\mathrm{MHS},s}\, e^{-\epsilon[s+\frac{1}{2}(\mathrm{d}-4)]}\Big|_{\epsilon\to 0,\ \mathrm{fin.}} = 0$$

Remarks:

• proper-time cutoff for each s: power divergences Λ^n sum up to 0 too
(cf. supersymmetric theories)

• $\big(Z_{\rm MHS}\big)_{\rm tot} = 1$ need not apply to quotients of flat or $AdS_{\rm d}$ space
e.g. $Z_{\rm MHS}$ on thermal quotient of $AdS_{\rm d}$ is non-trivial

• conjecture: exact vacuum partition function of Vasiliev theory =1
i.e. $\big(Z_{\rm MHS}(AdS_{\rm d})\big)_{\rm tot} = 1$ to all orders in coupling
(analogy with supersymmetric or topological QFT)

• This is the consistency requirement of vectorial AdS/CFT duality:
log of partition function of dual free $U(N)$ scalar theory
has only $\mathcal{O}(N)$ term that should match classical action of Vasiliev theory
while all $g_{\rm HS} = 1/N$ corrections should be absent

Conformal higher spins

Flat space

free CHS field in d dimensions

$$S_s = \int d^d x \, \varphi_s P_s \partial^{2s+d-4} \varphi_s$$

P_s = projector to transverse traceless totally symmetric rank s field

Partition function in $d = 4$ $\quad (\Delta_s = -\partial^2)$ $\quad$ [AT 13]

$$Z_{\mathrm{CHS},s} = \Big[\frac{(\det \Delta_{s-1})^{s+1}}{(\det \Delta_s)^s} \Big]^{1/2} = \prod_{k=0}^{s-1} \Big[\frac{\det \Delta_{k\perp}}{\det \Delta_{s\perp}} \Big]^{1/2}$$

CHS fields of dim $\Delta = 2 - s$:

• sources or "shadow fields" for spin s conserved bilinear currents $J_s(\phi)$ built out of free $U(N)$ scalar field ϕ

• boundary values for the corresponding dual MHS theory in AdS_{d+1}

• interacting CHS theory may be defined as induced one [AT 02; Segal 02; Bekaert, Joung, Morad 10] integrating out ϕ in path integral with $\quad S = \int d^4 x \big[\partial\phi^* \partial\phi + \sum_s J_s(\phi) \varphi_s \big]$

• resulting interacting CHS theory contains all fields with spins $s = 0, 1, 2, ...$

Total free CHS partition function in flat background

$$(Z_{\rm CHS})_{\rm tot} = \prod_{s=1}^{\infty} Z_{{\rm CHS},s} = \Big[\frac{\det\Delta_0}{\det\Delta_1}\Big]^{1/2}\Big[\frac{(\det\Delta_1)^3}{(\det\Delta_2)^2}\Big]^{1/2}\Big[\frac{(\det\Delta_2)^4}{(\det\Delta_3)^3}\Big]^{1/2}\ldots$$

Formally cancelling similar factors

$$(Z_{\rm CHS})_{\rm tot} \;\; \to \;\; (Z_{\rm CHS})'_{\rm tot} = \prod_{s=0}^{\infty} \det\Delta_s$$

Alternative form (different regularization) $\quad (d=4)$:

$$Z_{{\rm CHS},s} = (Z_0)^{\nu_s} = [\det\Delta_0]^{-\nu_s/2}\,, \qquad\qquad \nu_s = s(s+1)$$

$$(Z_{\rm CHS})_{\rm tot} = \prod_{s=0}^{\infty}(Z_0)^{\nu_s} = (Z_0)^{\nu_{\rm tot}}\,, \qquad\qquad \nu_{\rm tot} = \sum_{s=0}^{\infty}\nu_s$$

Special regularization

$$\sum_{s=0}^{\infty} F(s) \;\; \to \;\; \sum_{s=0}^{\infty} F(s)\, e^{-\epsilon(s+\frac{d-3}{2})}\Big|_{\rm fin.}$$

• implied by relation to MHS theory in AdS_{d+1}

• should be the one that is consistent with symmetries of CHS theory

• implies vanishing of conformal a-anomaly [Giombi, Klebanov13; AT 13]

• implies vanishing of total number of CHS d.o.f. (e.g. in $d = 4$)

$$\nu_{\rm tot} = \sum_{s=0}^{\infty} s(s+1)\, e^{-\epsilon(s+\frac{1}{2})}\Big|_{\rm fin.} = 0\,, \qquad \text{i.e.} \qquad (Z_{\rm CHS})_{\rm tot} = 1$$

in general d even dimensions [AT 13]

$$Z_{{\rm CHS},s} = \Big[\big(\frac{1}{\det\Delta_{s\perp}}\big)^{\frac{d-4}{2}} \prod_{k=0}^{s-1} \frac{\det\Delta_{k\perp}}{\det\Delta_{s\perp}}\Big]^{1/2} = (Z_0)^{\nu_s}$$

$$\nu_s = \tfrac{(d-3)(2s+d-4)(2s+d-2)(s+d-4)!}{2(d-2)!\ s!}$$

$$\nu_{\rm tot} = \sum_{s=0}^{\infty} \nu_s\, e^{-\epsilon(s+\frac{d-3}{2})}\Big|_{\rm fin.} = 0 \qquad \to \qquad (Z_{\rm CHS})_{\rm tot} = 1$$

• ambiguity of regularization: if start with rearranged form + same cutoff

$$(Z_{\rm CHS})'_{\rm tot} = (Z_0)^{-2N_{\rm tot}}\ , \qquad N_{\rm tot} = \textstyle\sum_{s=0}^{\infty}(s+1)^2 = \frac{1}{24} \neq 0$$

Ricci-flat space

- CHS theory expected to admit a Ricci-flat (in general Bach) solution
- each CHS field should have proper gauge invariance on such space
- free CHS partition function in $R_{mn} = 0$ background

should be well-defined, i.e. gauge-independent

- Conjecture: CHS operator $\nabla^{2s} + \dots$ can be factorized

into product of s 2-derivative spin s Lichnerowicz operators

$$Z_{\mathrm{CHS},s} = \Big[\frac{(\det \Delta_{\mathrm{L}\, s-1})^{s+1}}{(\det \Delta_{\mathrm{L}\, s})^{s}}\Big]^{1/2}$$

$$\Delta_{\mathrm{L}\, s} = -\nabla_s^2 + X_s\ , \qquad (X_s\, \varphi)^{\mu_1\cdots\mu_s} = -s(s-1)\, R_{\nu}{}^{(\mu_1}{}_{\lambda}{}^{\mu_2}\varphi^{\mu_3\cdots\mu_s)\nu\lambda}$$

known to be true for $s = 1, 2$ but not for $s = 3$ [Nutma, Taronna 14]

ok if ignore $D_m R_{nklp}$ terms (not giving conformal anomaly in $d = 4$)

- then same rearrangement is possible:

$$(Z_{\mathrm{CHS}})_{\mathrm{tot}} = \prod_{s=1}^{\infty} Z_{\mathrm{CHS},s} \ \to \ (Z_{\mathrm{CHS}})'_{\mathrm{tot}} = \prod_{s=0}^{\infty} \det \Delta_{\mathrm{L}\, s}$$

• conformal anomaly [AT 13]

$$T_m^m = -\mathrm{a} R^* R^* + \mathrm{c}\, C^2 = (\mathrm{a}-\mathrm{c}) R^* R^* + 2\mathrm{c}\, W , \qquad W = R_{mn}^2 - \tfrac{1}{3} R^2$$

$$\mathrm{c}_s - \mathrm{a}_s = \tfrac{1}{720} \nu_s (4 - 45\nu_s + 15\nu_s^2) , \qquad \nu_s = s(s+1)$$

• if use same regularization $\sum_{s=0}^{\infty} F(s)\, e^{-\epsilon(s+\frac{d-3}{2})}$

$$\sum_{s=1}^{\infty} (\mathrm{c}_s - \mathrm{a}_s) = 0$$

while from computation on S^4

$$\sum_{s=1}^{\infty} \mathrm{a}_s = 0$$

implies 1-loop quantum consistency of CHS theory

Conformally-flat space: S^4

• no conformal anomaly – $Z_{\rm tot}$ of CHS theory on conformally-flat space simply related to one in flat space?

$Z_{\rm tot}(S^4)$ in same regularization is again =1? Yes!

• consistent with relation to massless HS partition function in AdS_5 [Giombi et al 13; AT 13; Becaria, AT 14] (also [Barvinsky 05,14])

$$Z_{{\rm CHS},s}(S^4) = \frac{Z^-_{{\rm MHS},s}(AdS_5)}{Z^+_{{\rm MHS},s}(AdS_5)}$$

Indeed, $(Z^{\pm}_{\rm MHS}(AdS_5))_{\rm tot} = 1$ [Giombi, Klebanov, Safdi 14]

$\zeta_{\rm MHS}(0) = 0$ automatic in AdS_5 and $\zeta'_{\rm MHS}(0) = 0$ for both D and N b.c.

• subtle issue of coordinating UV regularization in S^4 with IR in AdS_5

• verified for leading $\log \Lambda$ term on both sides and then for full result in IR reg. where AdS_5 volume factorizes

• same in systematic dimensional regularization [Dorn, Diaz 07]: non-trivial transcendental parts match (sum to 0) [Beccaria, AT 15]

$$Z_{\mathrm{CHS},s}(S^4) = \prod_{k=0}^{s-1} Z_{s,k}$$

$$Z_{s,k} = \Big[\frac{\det \Delta_{k\,\perp}(M^2_{k,s})}{\det \Delta_{s\,\perp}(M^2_{s,k})}\Big]^{1/2}\ , \qquad M^2_{s,k}\Big|_{d=4} = s - (k-1)(k+2)$$

$$\mathrm{F_s} = -\ln Z_{\mathrm{CHS},s}(S^4) = 4\mathrm{a}_s \ln \Lambda + F_s$$

$$\mathrm{a}_s = \tfrac{1}{720} s^2(s+1)^2(14s^2+14s+3)$$

$$F_s = -\tfrac{1}{6}s(s+1)(5s^2+5s+1)\ \ln A - \tfrac{1}{3}s(s+1)\,\zeta'(-3) + \dots$$

A=Glaisher constant=$-\frac{1}{2\pi^2}\zeta'(2) + \frac{1}{12}\ln(2\pi) + \frac{1}{12}\gamma_{\mathrm{E}}$

$\ln A$ and $\zeta'(-3)$ terms match similar terms on AdS_5 side

coeffs. sum to 0: $\sum_{s=1}^{\infty} \mathrm{a}_s$=0, ..., in same regularization of $\sum_s$

$$\big(Z_{\mathrm{CHS}}\big)_{\mathrm{tot}}(S^4) = \prod_{s=1}^{\infty} Z_{\mathrm{CHS},s}(S^4) = 1$$

Conformal symmetric tensor theory

Generic (non)unitary free conformal field in $d = 4$:

$(\Delta; j_1, j_2)$ of $SO(2,4)$ vs corresponding dual field in AdS_5:

general formulae for partition functions on S^4, $S^3 \times S^1$;

conformal anomaly coefficients a, c

Aim: test general expressions on

non-trivial example – $(1; \frac{s}{2}, \frac{s}{2})$:

conformal symmetric tensor (CST)

- described by 2-derivative action
- Weyl-invariant in curved background like CHS field
- lacks proper gauge invariance of massless HS (non-unitary):

only scalar gauge invariance in conf-flat space:

"maximal depth" – minimal gauge invariance– representative

of family of conformal fields [Bekaert, Grigoriev 13]

[CHS dual to MHS in AdS_5 is maximal gauge invariance member]

- corresponds to "maximal depth" partially massless field in AdS_5

Generalized "triple" ($d = 4$): [Bekaert, Grigoriev 13]

- higher-order conformal scalar operators in R^4 (higher-order singletons)

$$S = \int d^4x \, \phi_i^* (\partial^2)^\ell \phi_i$$

$\rightarrow$ partially conserved currents of spin s and depth t, $1 \leqslant t \leqslant s$:

$$\partial_{m_1 \ldots m_t} J^{m_1 \ldots m_s} = 0$$

Verma module $V(\Delta, s) = V(3 + s - t, s)$ reducible for $t \leqslant s$

irreducible module $D(3 + s - t, s) = V(3 + s - t, s)/V(3 + s, s - t)$

[Dolan, Nappi, Witten 01; Shaynkman, Typunin, Vasiliev 04]

- sources for currents or "shadow" fields – primary conformal depth t fields:

totally symmetric traceless $\varphi_{m_1 \ldots m_s}$ of dim $\Delta = 1 + t - s$

with $(\partial^2)^{1+s-t}$ action

- partially massless fields in AdS_5 with ∇^2 action

$$\delta\phi_{\mu_1 .. \mu_s} = \nabla_{\mu_1} ... \nabla_{\mu_t} \epsilon_{\mu_{t+1} \ldots \mu_s} + (g_{\mu\nu} - \text{terms})$$

dual to $J_{m_1 \ldots m_s}$ or have $\varphi_{m_1 \ldots m_s}$ as boundary values

- minimal depth case $t = 1$: ∂^2 scalar, conserved currents, CHS, MHS
- maximal depth case $t = s$: CST instead of CHS

field content of dual to $(\partial^2)^\ell$ scalar theory:
generalized Flato-Fronsdal theorem [Bekaert, Grigoriev 13]

$$D(2-\ell,0)\otimes D(2-\ell,0)=\oplus_{s=0}^{\infty}\oplus_{k=1}^{\ell}D(4+s-2k,s)$$

sum over PM fields of different odd depths $t=1,...,2\ell-1$

maximal depth $t=s$ fields do not form "closed subset"
in contrast to MHS in minimal depth case

Weyl invariant action for totally symmetric traceless tensor φ_s

$$\int d^d x \sqrt{g}\, \varphi_s (\nabla^2)^n \varphi_s + ... , \qquad n = 1, 2, ...$$

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu} , \qquad \varphi'_{\mu_1 \ldots \mu_s} = \Omega^\gamma \varphi_{\mu_1 \ldots \mu_s} , \qquad \gamma = s + n - \tfrac{1}{2} d$$

conformal operators with $n = 1, 2, ...$ generalise scalar $s = 0$ GJMS family

• CHS case: $n = s + \frac{1}{2}(d - 4)$ and $\gamma = 2s - 2$

• CST case: $n = 1$ and $\gamma = s - \frac{1}{2}(d - 2)$

Conformally flat space:

• larger n – more gauge symmetries consistent with locality of action:

CHS case maximal gauge symmetry with rank $s - 1$ tensor parameter

CST $n = 1$ case – only $\delta\varphi_s = \partial^s \sigma$ scalar gauge symmetry

• less than maximal gauge symmetry:

no unitary gauge to eliminate time-like components

• 2-derivative Fronsdal massless (maximally gauge-invariant) HS

is unitary but not conformally invariant

while $n = 1$ CST is conformally invariant but not unitary

Weyl-covariant CST $n=1$ Lagragrangian in d dim [Erdmenger, Osborn 97]

$$\mathcal{L}_s(d) = \nabla^\lambda \varphi^{\mu_1\cdots\mu_s} \nabla_\lambda \varphi_{\mu_1\cdots\mu_s} - \tfrac{4s}{2s+d-2}\, \nabla_\rho \varphi^{\rho\mu_1\cdots\mu_{s-1}} \nabla^\lambda \varphi_{\lambda\mu_1\cdots\mu_{s-1}}$$
$$+ \tfrac{2s}{d-2} R_{\rho\lambda} \varphi^{\rho\mu_1\cdots\mu_{s-1}} \varphi^\lambda{}_{\mu_1\cdots\mu_{s-1}} - \tfrac{4s-d^2+4d-4}{4(d-1)(d-2)}\, R\, \varphi^{\mu_1\cdots\mu_s} \varphi_{\mu_1\cdots\mu_s}$$
$$+ \omega\, C_{\alpha\beta\rho\lambda}\, \varphi^{\alpha\rho\mu_1\cdots\mu_{s-2}}\, \varphi^{\beta\lambda}{}_{\mu_1\cdots\mu_{s-2}}$$

C is Weyl tensor and ω is arbitrary const

$d=4$ case: flat background: $\quad \delta\varphi_{\mu_1\cdots\mu_s} = \partial_{\{\mu_1}...\partial_{\mu_s\}}\sigma$

$$\mathcal{L}_{\mathrm{CST},s} = \partial^\lambda \varphi^{\mu_1\cdots\mu_s} \partial_\lambda \varphi_{\mu_1\cdots\mu_s} - \tfrac{2s}{s+1} (\partial^\lambda \varphi_{\mu_1\cdots\mu_{s-1}\lambda})^2$$

φ_s: $\quad (1; \frac{s}{2}, \frac{s}{2})$ representation of $SO(2,4)$

unitary ($\Delta \geqslant 2 + j_1 + j_2$) only for $s = 0, 1$ (scalar and Maxwell)

gauge parameter σ is in representation $(1-s; 0, 0)$

thus CST describes “short” representation

$$[1; \tfrac{s}{2}, \tfrac{s}{2}] = (1; \tfrac{s}{2}, \tfrac{s}{2}) - (1-s; 0, 0)$$

(particular degenerate module of conformal group

[Shaynkman, Typunin, Vasiliev 04; Bekaert, Grigoriev 13])

$s=2,\ d=4$: "conformal spin 2" (not to confuse with PM field)

$$\mathcal{L}_{\rm CST,2} = \nabla^\lambda\varphi^{\mu\nu}\,\nabla_\lambda\varphi_{\mu\nu} - \tfrac{4}{3}\,(\nabla_\mu\varphi^{\mu\nu})^2 + 2\,R_{\rho\lambda}\,\varphi^{\mu\rho}\,\varphi_\mu^{\ \lambda}$$
$$- \tfrac{1}{6}\,R\,\varphi^{\mu\nu}\,\varphi_{\mu\nu} + \omega\,C_{\mu\nu\rho\lambda}\,\varphi^{\mu\rho}\,\varphi^{\nu\lambda}\,, \qquad \varphi_{\mu\nu}g^{\mu\nu}=0$$

compare: Einstein graviton in generic background

$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu},\ \varphi_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}h,\ h = h^\mu_\mu$

$$\mathcal{L}_{\rm E} = \nabla^\lambda\varphi^{\mu\nu}\,\nabla_\lambda\varphi_{\mu\nu} - 2\left[\nabla_\mu(\varphi^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}\,h)\right]^2 + \tfrac{1}{4}h\nabla^2 h$$
$$+ \tfrac{5}{3}\,R\,\varphi^{\mu\nu}\,\varphi_{\mu\nu} - 2\,C_{\mu\alpha\nu\beta}\,\varphi^{\mu\nu}\,\varphi^{\alpha\beta}$$

with standard (vector-parameter) gauge invariance on $R_{\mu\nu}=0$ backgr.

• in contrast to massless spin 0 and spin 1 Einstein graviton
does not represent conformal theory in flat space [Grishchuk 80]

• scale invariance but no special conformal invariance for all
2-derivative massless higher spin $s \geqslant 2$ fields in flat space

cf. Partially massless spin 2:

dS_d or AdS_d background: $R = \frac{2d}{d-2}\Lambda$

$$L = \nabla^\lambda h^{\mu\nu}\,\nabla_\lambda h_{\mu\nu} - 2(\nabla_\mu h^{\mu\nu})^2 + 2\nabla_\mu h^{\mu\nu}\nabla_\nu h - \nabla^\mu h \nabla_\mu h$$
$$+ \tfrac{1}{d}R(h^{\mu\nu}h_{\mu\nu} - \tfrac{1}{2}h^2) - \tfrac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} - h^2)$$
$$m^2 = \tfrac{d-2}{d(d-1)}R\,, \qquad \delta h_{\mu\nu} = (\nabla_\mu\nabla_\nu + \tfrac{m^2}{d-2}g_{\mu\nu})\epsilon$$

• flat space: conformally inv. e.o.m. [Drew, Gegenber 80; Barut, Xu 82]

$$\mathcal{L}_{\rm CST,2} = \partial^\lambda \varphi^{\mu\nu}\, \partial_\lambda \varphi_{\mu\nu} - \tfrac{4}{3}\, (\partial_\mu \varphi^{\mu\nu})^2 \,, \qquad \delta\varphi_{\mu\nu} = \big(\partial_\mu\partial_\nu - \tfrac{1}{4}\, g_{\mu\nu}\, \partial^2\big)\, \sigma$$
$$\partial^2 \varphi_{\mu\nu} - \tfrac{4}{3}\big(\partial_\alpha \partial_{(\mu}\, \varphi^\alpha_{\nu)} - \tfrac{1}{4}\, g_{\mu\nu}\, \partial_\alpha\partial_\beta\, \varphi^{\alpha\beta}\big) = 0$$

$\varphi_{\mu\nu}$: $(\Delta; j_1, j_2) = (1;1,1)$ representation of $SO(2,4)$
non-unitary [Fang, Heidenrich, Xu 83]

• describes combination of spin 2 and two spin 1 massless on-shell fields with $(9-1)-2 = 6$ physical d.o.f.

$$\varphi_{\mu\nu} = \varphi^\perp_{\mu\nu} + \partial_{(\mu}\, V^\perp_{\nu)} + \big(\partial_\mu\partial_\nu - \tfrac{1}{4}\, g_{\mu\nu}\, \partial^2\big)\, \sigma$$

• flat-space partition function

$$Z_{\rm CST,2} = \Big[\frac{(\det \Delta_0)^3}{\det \Delta_2}\Big]^{1/2} = \Big[\frac{\det \Delta_{1\perp}}{\det \Delta_{2\perp}}\Big]^{1/2} \frac{\det \Delta_0}{\det \Delta_{1\perp}} = Z_2 (Z_1)^2$$

• attempts of curved space generalizations:
on eqs of motion, with $\nabla^\mu\, \varphi_{\mu\nu} = 0$, non-Lagrangian [Grischuk, Yudin 80]

• consistent $d=4$ Lagrangian:

$\omega=0$ [Deser, Nepomechie 83]; $\omega=-2$ version [Leonovich 84]

general form with arbitrary ω [Erdmenger, Osborn 97]

• scalar gauge invariance present in conf. flat space

extends to Einstein background $\delta\varphi_{\mu\nu}=\left(\nabla_\mu\nabla_\nu-\frac{1}{4}g_{\mu\nu}\nabla^2\right)\sigma$

provided $\omega=-2$ (as in Lichnerowitz operator);

suggests possibility of consistent coupling to Einstein gravity

• $\omega=-2$ case is special: for $R_{\mu\nu}=0$

kinetic operator factorizes on spin 2 and 1 parts (i.e. becomes diagonal)

$$\begin{aligned}\mathcal{L}_{\rm CST,2}&=\nabla^\lambda\varphi^{\mu\nu}\nabla_\lambda\varphi_{\mu\nu}-\tfrac{4}{3}(\nabla_\mu\varphi^{\mu\nu})^2-2\,C_{\mu\nu\rho\lambda}\varphi^{\mu\rho}\varphi^{\nu\lambda}\\&=\varphi^\perp_{\mu\nu}\Delta_{\rm L\,2}\varphi^{\perp\,\mu\nu}+\tfrac{2}{3}V^\perp_\mu(\Delta_{\rm L\,1})^2V^{\perp\,\mu}\end{aligned}$$

where $\varphi_{\mu\nu}=\varphi^\perp_{\mu\nu}+\nabla_{(\mu}V^\perp_{\nu)}+\left(\nabla_\mu\nabla_\nu-\frac{1}{4}g_{\mu\nu}\nabla^2\right)\sigma$

$$Z_{\rm CST,2}=\Big[\frac{(\det\Delta_0)^2}{\det\Delta_{\rm L\,2\,\perp}\det\Delta_{\rm L\,1\,\perp}}\Big]^{1/2}=\Big[\frac{(\det\Delta_0)^3}{\det\Delta_{\rm L\,2}}\Big]^{1/2}$$

$(\Delta_{\rm L\,2})_{\mu\nu,\alpha\beta}=-g_{\mu(\alpha}g_{\beta)\nu}\nabla^2-2C_{\mu\alpha\nu\beta},\ (\Delta_{\rm L\,1})_{\mu\nu}=-g_{\mu\nu}\nabla^2,\ \Delta_0=-\nabla^2$

Partition function on S^4

unit-radius S^4, i.e. $R = 12$, $\Delta_{s\perp}(M^2) = -\nabla^2 + M^2$

$$\mathcal{L}_{\rm CST,2}(S^4) = \varphi^{\perp}_{\mu\nu}\,\Delta_{2\perp}(4)\,\varphi^{\perp\,\mu\nu} + \tfrac{2}{3}\,V^{\perp}_{\mu}\,\Delta_{1\perp}(3)\Delta_{1\perp}(-3)\,V^{\perp\,\mu}$$

$$Z_{\rm CST,2} = Z_{2,0}\,Z_{1,0}\;, \quad Z_{1,0} = \Big[\frac{\det\Delta_0(0)}{\det\Delta_{1\perp}(3)}\Big]^{1/2}\,, \quad Z_{2,0} = \Big[\frac{\det\Delta_0(-4)}{\det\Delta_{2\perp}(4)}\Big]^{1/2}$$

$Z_{1,0}$= Maxwell partition function

$Z_{2,0}$= partition function of $s = 2$ partially massless field

• Compare to $Z_{\rm CHS,2} = Z_{2,1}Z_{2,0}$ = Weyl graviton partition function

$Z_{2,1}$ = Einstein graviton partition function $= \big[\frac{\det\Delta_{1\perp}(-3)}{\det\Delta_{2\perp}(2)}\big]^{1/2}$

$$Z_{\rm CST,2} = \frac{Z_{\rm CHS,2}\;Z_{1,0}}{Z_{2,1}}$$

• partition function on $S^1_\beta \times S^3$ has interpretation

in terms of counting of conformal operators in spin 2 CFT in $\mathbb{R}^4$

for "shortened" representation with shadow counterpart $(3;1,1) - (5;0,0)$

Conformal anomaly coefficients

$d = 4$ conformal anomaly

$$T = \beta_1 R^* R^* + \beta_2 \big(R^2_{\mu\nu} - \tfrac{1}{3} R^2\big) = -\mathrm{a}\, R^* R^* + \mathrm{c}\, C^2 \ , \quad \beta_1 = \mathrm{c} - \mathrm{a} \ , \ \ \beta_2 = 2\mathrm{c}$$

can be found by two separate computations:

(i) on conformally-flat space, e.g., S^4 – a coefficient

(ii) on a Ricci-flat space – c-a coefficient

a: follows from S^4 partition function (does not depend on ω)

$$\mathrm{a}[\Delta_{\perp\, s}(M^2)] = \tfrac{1}{720}(2s+1)\big[30s^3+85s^2+10s-58-30(s^2-2)M^2-15M^4\big]$$

$$\mathrm{a}_{\rm CST,2} = \mathrm{a}_{2,0} + \mathrm{a}_{1,0} = \tfrac{53}{45} + \tfrac{31}{180} = \tfrac{27}{20}$$

c: Z is non-trivial for generic ω – non-diagonal $K^{\mu\nu}\nabla_\mu\nabla_\nu + \ldots$

simplifies for $\omega = -2$: then from Z on $R_{\mu\nu} = 0$ background

using $\beta_1[\Delta_{\mathrm{L}\, s}] = \tfrac{1}{720}(s+1)^2\big[21 - 20(s+1)^2 + 3(s+1)^4\big]$

$$(\mathrm{c}-\mathrm{a})_{\rm CST,2} = \tfrac{31}{30} \ , \qquad \text{i.e.} \qquad \mathrm{c}_{\rm CST,2} = \tfrac{143}{60}$$

AdS/CFT relation

Aim: check general expressions for conformal anomaly coefficients of massive $SO(2,4)$ representations found using [Beccaria, AT 14]

$$Z(\partial M^5) = \frac{Z^-(M^5)}{Z^+(M^5)}\,, \qquad \mathcal{A} = \mathcal{A}^- - \mathcal{A}^+ = -2\mathcal{A}^+$$

• 4d conformal field $(\Delta'; j_1, j_2)$: corresponding shadow field $(\Delta; j_1, j_2)$ with $\Delta = 4 - \Delta'$ is associated (as a boundary value) to massive higher spin in AdS_5 with mass $m^2 = (\Delta - 2)^2 - s^2,\ \ s = j_1 + j_2$ (kinetic operator for $j_1 \geqslant j_2$ is $-\nabla^2 + \Delta(\Delta - 4) - 2j_1$ [Metsaev 03])

• massive field in long rep. $(\Delta; \frac{s}{2}, \frac{s}{2})$: get via AdS_5 [Giombi et al 13]

$$\widehat{\mathrm{a}}(\Delta; \tfrac{s}{2}, \tfrac{s}{2}) = \tfrac{1}{720}(s+1)^2(\Delta-2)^3\big(-3\Delta^2 + 12\Delta + 5s^2 + 10s - 7\big)$$

• non-unitary CST 4d field $[1;1,1] = (1;1,1) - (-1;0,0)$

$\to$ "partially massless" $[3;1,1] = (3;1,1) - (5;0,0)$ spin 2 in AdS_5

($\Delta' = 1$ for rank 2 field and $\Delta' = -1$ for scalar gauge parameter)

$$\mathrm{a}_{\mathrm{CST},2} = \widehat{\mathrm{a}}(3;1,1) - \widehat{\mathrm{a}}(5;0,0) = \tfrac{27}{20}$$

• $\mathrm{c}-\mathrm{a}$ for field associated to $(\Delta; \frac{s}{2}, \frac{s}{2})$ massive field in AdS_5
[Beccaria, AT 14] (cf. [Mansfield et al 03, Ardehali et al 13])

$$(\widehat{\mathrm{c}}-\widehat{\mathrm{a}})(\Delta; \tfrac{s}{2}, \tfrac{s}{2}) = \tfrac{1}{720}(s+1)^2(\Delta-2)\Big[-3(\Delta-2)^4 - 5(s^2+2s-3)(\Delta-2)^2 + 8s^3 + 2s^2 - 12s - 8\Big]$$

$$\mathrm{c}_{\mathrm{CST},2} = \widehat{\mathrm{c}}(3;1,1) - \widehat{\mathrm{c}}(5;0,0) = \tfrac{143}{60}$$

Agreement with direct computations in 4d

Conformal symmetric tensor of rank s

$\mathcal{L}_{\mathrm{CST},s}$ in $d=4$ for $s>2$:

• flat-space scalar gauge invariance survives in curved space
only in conformally-flat case $C_{\mu\nu\lambda\rho}=0$
e.g. absent for $s>2$ in Ricci flat background for any value of ω
(cf. analogy with standard massless $s>2$ HS with ∂^2 action)

• Weyl-covariant kinetic operator does not factorize (non-minimal) if $C\neq 0$

• flat space:

$$Z_{\mathrm{CST},s}=\Big[\frac{(\det\Delta_0)^{s+1}}{\det\Delta_s}\Big]^{1/2}=\prod_{k=1}^{s}\Big[\frac{\det\Delta_0}{\det\Delta_{k\,\perp}}\Big]^{1/2}=(Z_0)^{\nu_s}$$

$$\nu_s=N_s-(s+1)=\big[\tfrac{(2s+d-2)(s+d-3)!}{(d-2)!s!}-s-1\big]_{d=4}=s(s+1)$$

• same number of d.o.f. as for CHS field of same spin:
flat-space partition functions match, i.e. (in same regularization)

$$\big(Z_{\mathrm{CST}}\big)_{\mathrm{tot}}\equiv\prod_{s=1}^{\infty}Z_{\mathrm{CST},s}=\big(Z_{\mathrm{CHS}}\big)_{\mathrm{tot}}=1$$

• counting conformal gauge-invariant operators $\to \mathcal{Z}(q) = \sum_r \mathrm{d}_r \, q^{\Delta_r}$
in agreement with partition function on conformally-flat $S^1_\beta \times S^3$

$$Z_{\mathrm{CST},s}(S^1_\beta \times S^3) = \prod_{k=1}^{s} \Big[\frac{1}{(\det \mathbf{\Delta}_{k\,\perp})^{s-k+1}} \Big]^{1/2} = \exp \sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}_{\mathrm{CST},s}(q^m)$$

$q = e^{-\beta}$ and $\mathbf{\Delta}_{k\,\perp}$ defined on 3d tensors in decomposition of φ_s
single-particle partition function

$$\mathcal{Z}_{\mathrm{CST},s}(q) = \sum_{k=1}^{s} \sum_{r=1}^{s-k+1} \sum_{n=r-1}^{\infty} 2\,(n+1)\,(n+2k+1)\, q^{n+s-2r+3}$$
$$= \tfrac{2q^2 [q^{s+1} - (s+1)^2\, q + s(s+2)]}{(1-q)^4} = \tfrac{2q^2 [s(s+2) - q - q^2 - \ldots - q^s]}{(1-q)^3}$$

• partition function on S^4:

$$Z_{\mathrm{CST},s}(S^4) = \prod_{k=1}^{s} Z_{k,0}\ , \qquad\qquad Z_{k,0} = \Big[\frac{\det \Delta_0(M^2_{0,k})}{\det \Delta_{k\,\perp}(M^2_{k,0})} \Big]^{1/2}$$
$$M^2_{k,0} = 2 + k\ , \qquad\qquad M^2_{0,k} = 2 - k - k^2$$

• a-coefficient of conformal anomaly

$$\mathrm{a}_{\mathrm{CST},s} = \tfrac{1}{720} s(s+1)^2(3s^2+14s+14)$$

$\mathrm{a}_{\mathrm{tot}} = \sum_{s=1}^{\infty} \mathrm{a}_{\mathrm{CST},s} \neq 0$ does not vanish in any natural regularization
but no a priori reason why one needs to sum over all s here
in MHS and CHS cases summation was implied, in particular,
by AdS/CFT duality and relation to conserved currents of boundary theory
• c-coefficient: computation on Ricci-flat space is problematic:
(i) lack of scalar gauge inv. – partition function is gauge dependent
(ii) the non-minimal nature (lack of factorization) of kinetic operator
need more complicated methods [Moss, Toms 13; Barvinsky,Vilkovisky 88]
ignoring these problems (as in CHS case this may not affect c)

$$\widetilde{Z}_{\mathrm{CST},s} = \Big[\frac{(\det \Delta_0)^{s+1}}{\det \Delta_{\mathrm{L}\, s}}\Big]^{1/2}$$

$$(\mathrm{c}-\mathrm{a})_{\mathrm{CST},s} = \tfrac{1}{720} s(s+1)(3s^4+15s^3+10s^2-30s-24)$$

AdS_5 connection

• CST field: $[1,\frac{s}{2},\frac{s}{2}]=(1,\frac{s}{2},\frac{s}{2})-(1-s,0,0)$

associated to spin s field in AdS_5 (shadow representation)

$[3,\frac{s}{2},\frac{s}{2}]=(3;\frac{s}{2},\frac{s}{2})-(3+s;0,0)$

"maximal-depth" partially massless field in AdS_5 [Bekaert, Grigoriev 13]

• one-particle $\mathcal{Z}$ of 4d CFT written in terms of AdS_5 partition functions or conformal characters [Beccaria, Bekaert, AT 14; Beccaria, AT 14]

$$\mathcal{Z}_{\mathrm{CST},s}(q)=\mathcal{Z}_s^-(q)-\mathcal{Z}_s^+(q)=\mathcal{Z}^+(q^{-1})-\mathcal{Z}^+(q)+\sigma(q)$$

$$\mathcal{Z}_s^+(q)=\widehat{\mathcal{Z}}^+(3;\ \tfrac{s}{2},\tfrac{s}{2})-\widehat{\mathcal{Z}}^+(s+3;\ 0,0)=\frac{(s+1)^2\,q^3-q^{s+3}}{(1-q)^4}$$

$$\widehat{\mathcal{Z}}^+(\Delta;j_1,j_2)=(2j_1+1)(2j_2+1)\frac{q^\Delta}{(1-q)^4}$$ is character of long rep.

"correction term" due to scalar gauge symmetry:

$\sigma(q)=\frac{1}{6}s(s+1)(s+2)+\frac{1}{6}\sum_{n=1}^{s-1}n(n+1)(n+2)(q^{n-s}+q^{s-n})$

• a-anomaly computed using AdS_5 relation

$$\mathrm{a}_{\mathrm{CST},s}=\widehat{\mathrm{a}}(3;\tfrac{s}{2},\tfrac{s}{2})-\widehat{\mathrm{a}}(3+s;0,0)=\tfrac{1}{720}s(s+1)^2(3s^2+14s+14)$$

• formal c expression also matches: $\mathrm{c}_{\mathrm{CST},s}=\mathrm{c}(3;\frac{s}{2},\frac{s}{2})-\mathrm{c}(3+s;0,0)$

Conclusions

• remarkable symmetries of (bosonic) higher spin theories
with all spins included:
one-loop vacuum $Z = 1$; zero effective number of d.o.f.;
cancellation of conformal anomalies (UV divergences)
• importance of regularization of sum over s consistent with symmetries:
crucial part of definition of quantum theory

• MHS and CHS families are special, but CST is not

• interesting example of "conformal spin 2" field:
non-unitary, related to "partially massless" field in AdS_5

• check of expressions for partition functions and conformal anomalies
for generic 4d conformal fields:
consistent with "kinematical" $\mathrm{AdS}_5/\mathrm{CFT}_4$