Remarks on Vasiliev's equations to the second order

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based on a work in collaboration with G.Lucena Gomez, P.Kessel, E.Skvortsov
arXiv:1505.05887 & in progress

Intro and Goals

3d HS theories can be considered as a toy model for their higher-dimensional relatives even though 4d Vasiliev's equations are even simpler than 3d ones. (see Zhenya's talk)

In this talk:

- Extract the scalar couplings from 3d Vasiliev's equations (4d: see Zhenya)
- Link the above couplings to an ordinary Lagrangian formulation

In the near future:

- Test Gaberdiel-Gopakumar conjecture beyond symmetry considerations
- Improve our understanding of non-localities (functional class) in Vasiliev's theory

$$J_{\underline{m}(s)} = \sum_l g_l \,\Box^l \left(\ldots + \nabla_{\underline{m}(s-k)} \Phi^* \nabla_{\underline{m}(k)} \Phi + \ldots \right)$$

*Any implicit reference is to Vasiliev's works

$$so(2,2) \sim sp(2) \oplus sp(2)$$
$$U(sp_2)/(C_2 - (\lambda^2 - 1))$$
$$hs(\lambda) \oplus hs(\lambda)$$

Ingredients

$$\omega_{\underline{m}}^{a(s-1)} = \epsilon^a{}_{bc}\, \omega_{\underline{m}}^{a(s-2)b,c}$$

- The frame-like formalism in 3d deals with (Vasiliev's 1980):

$$e_{\underline{m}}^{a(s-1)} \qquad \omega_{\underline{m}}^{a(s-1),b} \qquad \dots \qquad \omega^{a(s-1),b(k)}$$

- 3d isomorphism: $so(2,2) \sim sp(2) \oplus sp(2)$

The HS algebra (λ=1/2) is conveniently formulated in the sp(2) language (Vasiliev 1992)

$$[L_{\alpha\alpha}, L_{\beta\beta}] = \epsilon_{\alpha\beta} L_{\alpha\beta}\,, \qquad [L_{\alpha\alpha}, P_{\beta\beta}] = \epsilon_{\alpha\beta} P_{\alpha\beta}\,, \qquad [P_{\alpha\alpha}, P_{\beta\beta}] = \epsilon_{\alpha\beta} L_{\alpha\beta}\,.$$

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta} \qquad \phi^2 = 1 \qquad \psi^2 = 1 \qquad \{\phi, \psi\} = 0$$

$$L_{\alpha\beta} = -\frac{i}{4}\{\hat{y}_\alpha, \hat{y}_\beta\} \qquad P_{\alpha\beta} = \phi L_{\alpha\beta}$$

$$f \in hs: \qquad f(\hat{y}, \phi) = f_0(\hat{y}, \phi) + \psi f_1(\hat{y}, \phi)$$

HS algebra and Free Fields

Vacuum solution = flat so(2,2) Connection of HS algebra

$$d\Omega = \Omega \star \Omega \longrightarrow \Omega = \frac{1}{2}\varpi^{\alpha\alpha}L_{\alpha\alpha} + \frac{1}{2}h^{\alpha\alpha}P_{\alpha\alpha}$$

Vacuum AdS_3 solution

Free equations have a plain algebraic meaning in terms of HS-algebra:

$$d\omega = \{\Omega, \omega\} \qquad\qquad d\mathbf{C} = [\Omega, \mathbf{C}]$$

The fields are 1-form and 0-form modules of the HS-algebra satisfying covariant constancy conditions

Physical and Twisted sectors

The covariant constancy conditions written before are reducible due to the absence of ψ in Ω:

$$[\phi f, g\psi] = \phi\{f, g\}\psi$$

One can then split fields according to whether or not they involve ψ

$$\boldsymbol{\omega} = \omega + \tilde{\omega}\psi \qquad \mathbf{C} = \tilde{C} + C\psi$$

$$\widetilde{D}\tilde{\omega} = 0 \qquad D\tilde{C} = 0$$

$$D\omega = 0 \qquad \widetilde{D}C = 0 \;\Rightarrow\; (\Box + \tfrac{3}{4})\Phi(x) = 0$$

- Gauge fields $\omega(y, \phi) \sim \phi e + \omega$
- A scalar (and spin ½ fermion) $C(\phi) + y^\alpha C_\alpha(\phi)$
- A constant + killing tensor fields $\tilde{C}(y, \phi)$
- twisted one forms leaving in infinite dimensional modules $\tilde{\omega}(y, \phi)$

$$D = \nabla - \frac{1}{2}h^{\alpha\alpha}[P_{\alpha\alpha}, \bullet]$$

$$\widetilde{D} = \nabla - \frac{1}{2}h^{\alpha\alpha}\{P_{\alpha\alpha}, \bullet\}$$

Unfolding and Interactions

From free theories to interacting theories Vasiliev's prescription is:

$$\begin{cases} d\boldsymbol{\omega} = F^{\omega}(\boldsymbol{\omega}, \mathbf{C}) \\ d\mathbf{C} = F^{C}(\boldsymbol{\omega}, \mathbf{C}) \end{cases}$$

$$d^2 = 0$$

$$F^{\omega}(\boldsymbol{\omega}, \mathbf{C}) = \mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}) + \mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}, \mathbf{C}) + \mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}, \mathbf{C}, \mathbf{C}) + \ldots$$

$$F^{C}(\boldsymbol{\omega}, \mathbf{C}) = \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}) + \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}, \mathbf{C}) + \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}, \mathbf{C}, \mathbf{C}) + \ldots$$

C-expansion

First cocycle governed by HS algebra (hs covariantization of free eqs.):

$$\mathcal{V}(\boldsymbol{\omega}, \boldsymbol{\omega}) = \boldsymbol{\omega} \star \boldsymbol{\omega} \qquad \mathcal{V}(\boldsymbol{\omega}, \mathbf{C}) = \boldsymbol{\omega} \star \mathbf{C} - \mathbf{C} \star \boldsymbol{\omega}$$

The system is endowed with fully non-linear gauge symmetries

$$\delta\boldsymbol{\omega} = d\boldsymbol{\xi} + \boldsymbol{\xi}\frac{\partial}{\partial\boldsymbol{\omega}}F^{\omega}(\boldsymbol{\omega}, \mathbf{C}) = d\boldsymbol{\xi} - [\boldsymbol{\omega}, \boldsymbol{\xi}]_{\star} + O(\mathbf{C}) \qquad \delta\mathbf{C} = \boldsymbol{\xi}\frac{\partial}{\partial\boldsymbol{\omega}}F^{C}(\boldsymbol{\omega}, \mathbf{C}) = \boldsymbol{\xi} \star \mathbf{C} - \mathbf{C} \star \boldsymbol{\xi} + O(\mathbf{C}^2)$$

Goal: Expand Vasiliev's equations and study the above cocycles to second order

Vasiliev's Equations

The cocycles are resummed in Vasiliev's equations with the help of an additional Z oscillator:

$$\mathcal{W} = \mathcal{W}_{\underline{m}}(y,z,\phi,\psi|x)\,dx^{\underline{m}} \qquad \mathcal{B} = \mathcal{B}(y,z,\phi,\psi|x) \qquad \mathcal{S}_\alpha = \mathcal{S}_\alpha(y,z,\phi,\psi|x)$$

$$f(y,z)\star g(y,z) = \frac{1}{(2\pi)^2}\int d^2u\,d^2v\;f(y+u,z+u)\,g(y+v,z-v)\exp\left(iv^\alpha u_\alpha\right)$$

$$\begin{aligned} d\mathcal{W} &= \mathcal{W}\star\mathcal{W} \\ d\mathcal{B} &= [\mathcal{W},\mathcal{B}] \\ d\mathcal{S}_\alpha &= [\mathcal{W},\mathcal{S}_\alpha] \\ 0 &= [\mathcal{B},\mathcal{S}_\alpha] \\ [\mathcal{S}_\alpha,\mathcal{S}_\beta] &= -2i\epsilon_{\alpha\beta}(1+\mathcal{B}) \end{aligned}$$

Vasiliev 1992

(well known) subtleties:

- Naive Lorentz generators fail beyond linear approximation
- The right Lorentz generators acquire non-linear corrections

$$\tfrac{1}{2}\omega^{\alpha\alpha}L_{\alpha\alpha} = \tfrac{1}{2}\omega^{\alpha\alpha}\left(-\frac{i}{2}(y_\alpha y_\alpha - z_\alpha z_\alpha) - \frac{i}{4}\{\mathcal{S}_\alpha,\mathcal{S}_\alpha\}\right)$$

Vasiliev

- Pseudo-local redefinition restores Lorentz

Puzzle for the AdS/CFT

In 3d Gaberdiel and Gopakumar argued that a scalar coupled to HS gauge sector is required but Killing tensors and twisted one forms have not been investigated from this perspective

Can one embed twisted one forms and killing tensors in AdS/CFT?

$$\begin{cases} \nabla \tilde{C}_{\pm}^{\alpha(n)} \mp h^{\alpha}{}_{\gamma}\, \tilde{C}_{\pm}^{\gamma\alpha(n-1)} = 0 \\ (\widetilde{D}\tilde{\omega}_{\pm})^{\alpha(2s)} = 0 \end{cases}$$

Since we do not understand this sector we will try to truncate the theory to only include gauge fields and scalars

About the possibility of truncating away the twisted sector we already know a yes go result to all orders (trivializing also physical fields)

→ Integration Flow --- Prokushkin-Vasiliev 1998

Can we truncate the twisted sector without touching the physical sector?

1st Order

$$\begin{cases} D\omega = 0\,, \\ \widetilde{D}\tilde{\omega} = \frac{1}{8}H^{\alpha\alpha}(y_\alpha + i\partial^w_\alpha)(y_\alpha + i\partial^w_\alpha)C(w,\phi)\Big|_{w=0} \\ D\tilde{C} = 0\,, \\ \widetilde{D}C = 0 \end{cases}$$

We search for a consistent truncation of the theory in which the twisted sector is not present (Vasiliev, 92)

$$\Delta\tilde{\omega} = \frac{1}{4}\phi h^{\alpha\alpha}\int_0^1 dt\ (t^2-1)(y_\alpha + it^{-1}\partial^y_\alpha)(y_\alpha + it^{-1}\partial^y_\alpha)C(yt,\phi)$$

- On top of the above we have an ambiguity given by $\mathbb{H}^1(\widetilde{D}, C) \neq \emptyset$

$$\Delta\tilde{\omega} = \frac{1}{4}\phi h^{\alpha\alpha}\int_0^1 dt\ (t^2-1)\Big[(g_0 y_\alpha y_\alpha + 2iy_\alpha t^{-1}\partial^y_\alpha - g_0 t^{-2}\partial^y_\alpha\partial^y_\alpha)C_{\text{bose}}(ty) + (y_\alpha y_\alpha + 2id_0 y_\alpha t^{-1}\partial^y_\alpha - t^{-2}\partial^y_\alpha\partial^y_\alpha)C_{\text{fermi}}(ty)\Big]$$

2^{nd} Order

The ambiguity in truncating the twisted sector induces an ambiguity at the level of second order interactions while we can set to zero all first order twisted fields

$$
\begin{cases}
\widetilde{D}C^{(2)} = \omega^{(1)} \star C^{(1)} - C^{(1)} \star \omega^{(1)}(-\phi) \\
D\widetilde{C}^{(2)} = \widetilde{\mathcal{V}}_{g_0,d_0}(\Omega, C^{(1)}, C^{(1)}) \\
\widetilde{D}\tilde{\omega}^{(2)} = \widetilde{\mathcal{V}}_{g_0,d_0,g_1,d_1}(\Omega, \omega^{(1)}, C^{(1)}) \\
D\omega^{(2)} = \omega^{(1)} \star \omega^{(1)} + \mathcal{V}_{g_0,d_0}(\Omega, \Omega, C^{(1)}, C^{(1)})
\end{cases}
$$

Cocycles depend on $g_0\,, d_0\,, g_1\,, d_1$

The twisted fields are sourced again at second order by physical fields!

The cohomology analysis allowed us to explicitly determine weather or not the backreaction on the twisted sector could or could not be removed

$$\mathcal{V}_{g_i,d_i} \neq D(\ldots)$$

$$
\begin{cases}
\mathbb{H}^1(D, CC) \neq \emptyset \\
\mathbb{H}^2(\widetilde{D}, \omega C) \neq \emptyset
\end{cases}
$$

Prokushkin-Vasiliev 1999

A uniqueness result

Result:

$$\left.\begin{aligned}&\widetilde{\mathcal{V}}_{g_i,d_i}(\Omega,\omega^{(1)},C^{(1)})\\&\widetilde{\mathcal{V}}_{g_0,d_0}(\Omega,C^{(1)},C^{(1)})\end{aligned}\right\}\quad \text{exact if:}\quad \boxed{\begin{gathered}g_0=d_0=g_1=d_1=\beta\\ \beta=0\end{gathered}}$$

- The ambiguity introduced at first order is fixed uniquely and completely by demanding the consistency of the truncation at the next order
- Few parameters kill infinitely many D-cohomologies. Not surprising -- after all those should arise from a single HS cohomolgy branched with respect to the AdS subalgebra
- The above point must coincide with Integration flow restricted to the twisted sector. However, here we are not redefining the physical sector

Generalized stress tensors

Having fixed the ambiguity in the second order theory and having truncated the twisted sector, we can now move to the physical backreaction (stress tensor) [Much more complicated than in 4d... See Zhenya's talk]

$$D\omega^{(2)} = \ldots + \mathcal{V}(\Omega, \Omega, C^{(1)}, C^{(1)}) \quad \Rightarrow \quad \Box \phi_{\underline{m}(s)} = J_{\underline{m}(s)}$$

$$J = (1 - \nabla Q^{-1})\mathcal{V}\Big|_{\phi=0}$$

The result takes the form:

$$\begin{cases} \mathcal{V}_{\alpha(2s)} = H_{\alpha\alpha} J'_{\alpha(2s-2)} + H_\alpha{}^\beta J^{\text{hook}}_{\beta\alpha(2s-1)} + H^{\beta\beta} J_{\beta\beta\alpha(2s)} \\ J_{\alpha(n+m)} = \sum_l a^{n,m,l} C_{\alpha(n)\nu(l)}(\phi) C^{\nu(l)}{}_{\alpha(m)}(-\phi) \end{cases}$$

$$a_l \sim \frac{1}{l^3 l!}$$

$$H^{\alpha\alpha} = h^\alpha{}_\gamma \wedge h^{\gamma\alpha}$$

$$C_{\alpha(2s)} \sim (\sigma^{\underline{m}}_{\alpha\alpha} \nabla_{\underline{m}})^s \Phi(x)$$

The expressions decompose in various independently conserved pieces but dressed with an infinite derivative tail

$$\sim \sum_l^\infty \Box^l (\Phi^* \nabla^s \Phi)$$

(Prokushkin-Vasiliev 1999)

Cohomology and backreaction

We can ask the same question we asked for the twisted sector backreaction

It was already noticed by Prokushkin-Vasiliev that canonical s-derivative currents are exact in cohomology

$$(\Phi^* \nabla^s \Phi) \sim D \left(\sum_l^\infty \Box^l \dots \right)$$

Projushkin-Vasiliev '99

Extending the analysis of Prokushkin-Vasiliev we find (expected from Integration Flow):

$$\mathbb{H}^2(D, CC) = \emptyset$$

Puzzle: physically we expect only to be able to relate pseudo-local backreactions to canonical currents --- functional class

Ordinary Lagrangian Formulation

We would also like to link the result obtained from Vasiliev's equations to a (perturbative) action principle.

$$S_{CS} = \frac{k}{4\pi}\int tr\left(\omega\wedge d\omega - \frac{2}{3}\omega\wedge\omega\wedge\omega\right)$$

$$S_{sc} = \int \det|h|\,\left(|\nabla\Phi|^2 + m^2|\Phi|^2\right)$$

$$S_{int} = \int tr\left[\omega(y,\phi)\star\wedge\mathbf{J}(y,\phi)\right] \sim \sum_s \frac{2g_s}{s}\int \phi_{\underline{m}(s)}j^{\underline{m}(s)}$$

$$S = S_{CS} + S_{sc} + S_{int}$$

The scalar field acquires some gauge transformations at this order

$$\delta^{(s)}\Phi = 2ig_s\,\xi_{\underline{m}(s-1)}(2i\nabla^{\underline{m}})^{s-1}\Phi \sim \mathrm{tr}\{\xi_e, C\}_\star$$

$$g_s = \frac{1}{(2s-2)!}$$

Local Cohomology

To have a better understanding of redefinitions it is useful to study local cohomologies

Idea:

$$\int \omega^{(s)}(\Box^l J) \sim \int (\Box^l \omega^{(s)}) J \sim C_l^{(s)} \int \omega^{(s)} J$$

In more detail in terms of frame-like formalism and covariant derivatives we want find coefficients C_l such that:

$$C_{2l}^{(s)} J^{can}_{\alpha(2s)} - \frac{1}{2l!} \Box^l J^{can}_{\alpha(2s)} = (DK)_{\alpha(2s)}$$

Local

Solution:

$$C_l^{(s)} = (-1)^l \frac{s\,(2s+l)!}{(2s)!(l+1)!} \Big[2\,(l+s)\,{}_2F_1(1, l+2s+1; l+2; -1) - l - 1 \Big] + 4^{-s}(l+s)$$

$$\boxed{C_l^{(s)} \sim l^{2s-1} \quad (l \to \infty)}$$

A Puzzle

As a very naive attempt one might combine the asymptotic behavior of the C-coefficients to formulate a convergence criterion after having independently integrated by parts and redefined away each box in the pseudo-local tail

$$C_l^{(s)} \sim l^{2s-1} \quad (l \to \infty) \qquad\qquad \sum a_l^{(s)} l! \, C_l^{(s)} = g_s$$

$$\Downarrow$$

$$a_l^{(s)} \prec \frac{1}{l^{2s}} \frac{1}{l!}$$

Puzzle: the backreaction we find does not pass this test for s>1

(Analitic continuation??)

$$a_l^{(s)} \sim \frac{1}{l^3} \frac{1}{l!}$$

Vasiliev Theory and Witten Diagrams

$$\langle \ldots \rangle \sim \int_{AdS_3} \mathrm{Tr}\left[\omega \star J\right] = \int_{AdS_3} \mathrm{Tr}\left[D\xi^{\mathbf{B.H.}} \star J\right] = \lim_{z\to 0} \int_{\partial AdS_3} \mathrm{Tr}\left[\boxed{\xi^{\mathbf{B.H.}}} \star J\right]$$

Brown-Henneaux boundary behavior

We have computed the Witten diagram associated to: $\Box^l J^{\mathrm{can.}}$

$$\left(\frac{z}{x^+x^- + z^2}\right)^{\lambda+s+l} \sim \sum_{k=0}^{s+l} \# z^{2-\lambda-s-l+2k} (\partial_{x^+}\partial_{x^-})^k \delta^{(2)}(x)$$

$$\boxed{\langle \ldots \rangle \sim \left(\sum_l a_l^{(s)} l! \, C_l^{(s)}\right) \frac{1}{|x_{12}|} \left(\frac{x_{12}^+}{x_{31}^+ x_{23}^+}\right)^s}$$

Puzzle: the naive field theory computation fails

$$\int_{AdS} \sum_l \overset{?}{\neq} \sum_l \int_{AdS}$$

Can we analytically continue the divergent sum?

Summary & Outlook

- Truncating the twisted sector in PV is a non-trivial task and requires a knowledge of HS cohomology order by order (see also Integration Flow) – puzzle for holography
- There exist other HS theory in d=3 without twisted sector. The d-dim theory at d=3 describes a HS theory w. hs$(\lambda=1)$ without the twisted sector from the start (while twisted field sources on physical fields are never quadratic). Generically the latter theory is expected to be different from PV (?)
- Understanding functional class is an important problem and would be a key element for a better understanding of the pseudo-local tails we see in the backreaction (see e.g. Vasiliev 2015). Physical way to check if a redefinition is allowed: compute observables (CFT correlators via Witten diagrams). Acceptable pseudo-local redefinitions should not have effect on those by definition. Puzzle! (in progress...)
- 4d theory is simpler and might have better convergence properties (see Zhenya's talk) [in progress...] and quartic is behind the corner!

The Cocycle

$$J^{PV} = J^{redef} + J^{phys} \quad J^{phys} = H^{\alpha\alpha}(J^Q_{\alpha\alpha} + J^P_{\alpha\alpha}) \quad J^{redef} = H^{\alpha\alpha}(J^K_{\alpha\alpha} + J^{R_1}_{\alpha\alpha} + J^{R_2}_{\alpha\alpha})$$

$$\begin{aligned} J^Q_{\alpha\alpha} =& \; \{d_1 y_\alpha y_\alpha + d_2 \xi_\alpha \xi_\alpha + d_3 \eta_\alpha \eta_\alpha + d_4 \xi_\alpha \eta_\alpha + d_5 \xi_\alpha y_\alpha + d_6 \eta_\alpha y_\alpha \\ & - d_7(y_\alpha \eta_\alpha (y\eta) - \xi_\alpha y_\alpha (y\xi) - \xi_\alpha \eta_\alpha (\xi\eta))\} Q \\ J^K_{\alpha\alpha} =& -\frac{1}{8} i (1-q)(1+t)((q+t)(\eta_\alpha \eta_\alpha - \xi_\alpha \xi_\alpha) - (q+1)(t+1) y_\alpha \eta_\alpha - (q-1)(t-1) y_\alpha \xi_\alpha \\ & + (1+qt) y_\alpha y_\alpha - (q-1)(t+1)(2qt - q + t) \xi_\alpha \eta_\alpha) K \\ & - \frac{1}{16} (q-1)^3 (q+1)(t-1)(t+1)^3 (\eta\xi) \xi_\alpha \eta_\alpha K \\ J^P_{\alpha\alpha} =& \; \{p_1 \xi_\alpha \xi_\alpha + p_2 \eta_\alpha \eta_\alpha + p_3(t\, \xi_\alpha \eta_\alpha + \xi_\alpha y_\alpha + \eta_\alpha y_\alpha)\} P \\ J^{R^1}_{\alpha\alpha} =& \; \{\rho_1 \xi_\alpha \xi_\alpha + \rho_2 \eta_\alpha \eta_\alpha + \rho_3 y_\alpha \xi_\alpha + \rho_4 y_\alpha \eta_\alpha + \rho_5 \xi_\alpha \eta_\alpha\} R^1 \\ & + \frac{1}{16} i \; (t^2 - 1) \{\xi_\alpha \eta_\alpha (t+2) - y_\alpha \eta_\alpha\} K_t \\ J^{R^2} =& -J^{R^1} \left(\begin{smallmatrix} t \to -q \\ q \to t \\ \xi \leftrightarrow \eta \end{smallmatrix} , \; \begin{smallmatrix} R^1 \to R^2 \\ K_t \to K_q \end{smallmatrix} \right) \end{aligned}$$

$$d_1 = \frac{i}{8}(-q + 4q^2 - 3q^3 + 4qt - 9q^2 t + 4q^3 t + 8q^2 t^2 + q^3 t^2)$$

...

5 exponent vs 1 in 4d:

$$Q = \exp i\,(tq(\eta + y)(y + \xi))$$

...

Spin-1 Backreaction

In the spin-1 case we find a pseudo-local source to the Maxwell tensor:

$$C_{\alpha(2s)} \sim (\sigma^{m}_{\alpha\alpha}\nabla_{\underline{m}})^{s}\Phi(x)$$

$$d\omega^{(2)} = H^{\beta\beta}\left[\sum_{l\in 2\mathbb{N}} a_l \left(C_{\beta\beta\nu(l)}(\phi)C^{\nu(l)}(-\phi) + C_{\nu(l)}(\phi)C^{\nu(l)}{}_{\beta\beta}(-\phi)\right) - \sum_{l\in 2\mathbb{N}+1} a_l\, C_{\beta\nu(l)}(\phi)C^{\nu(l)}_{\beta}(-\phi)\right]$$

$$H^{\alpha\alpha} = h^{\alpha}{}_{\gamma}\wedge h^{\gamma\alpha}$$

The coefficients are: ⟶ $$a_l = \frac{i(-i)^l}{l!}\frac{1}{(l+2)^2(l+4)}$$

The naive resummation is possible and one can remove all higher derivative tail:

$$a_0 = -\frac{i}{8}$$

Spin-2 Backreaction

To make contact with standard symmetric canonical current we need to solve torsion

We then obtain the manifestly symmetric backreaction:

$$R^{(2)}_{\alpha\alpha} = J^{\text{canonical}}_{\alpha\alpha} + J^{\text{Improvement}}_{\alpha\alpha}$$

$$H^{\alpha\alpha} = h^{\alpha}{}_{\gamma} \wedge h^{\gamma\alpha}$$

$$C_{\alpha(2s)} \sim (\sigma^{m}_{\alpha\alpha} \nabla_{\underline{m}})^{s} \Phi(x)$$

$$\left\{ \begin{array}{l} J^{\text{canonical}}_{\alpha\alpha} = H^{\beta\beta} J_{\beta\beta\alpha\alpha} \\ J_{\beta\beta\alpha\alpha} = \sum_{l \in 2\mathbb{N}} a_l \Big(C_{\alpha(4)\nu(l)}(\phi) C^{\nu(l)}(-\phi) + 3\, C_{\alpha(2)\nu(l)}(\phi) C^{\nu(l)}{}_{\alpha(2)}(-\phi) \Big) \end{array} \right.$$

$$a_l = \frac{i^{l-1}}{4\, l!} \left(\frac{1}{l+1} - \frac{6}{l+2} + \frac{9}{(l+3)^2} + \frac{19}{4(l+3)} - \frac{6}{l+4} + \frac{7}{l+5} - \frac{3}{4(l+7)} \right)$$

The resummation diverges (alternating series not absolutely convergent)

$$a_0 = -\frac{i}{24}$$

Spin-4 Backreaction

$$J_{\alpha(8)} = \sum_{l\in 2\mathbb{N}} a_l \left(C_{\alpha(8)\nu(l)}(\phi) C^{\nu(l)}(-\phi) + \ldots \right)$$

$$\begin{aligned}
a_l = & -\frac{i}{56l} + \frac{7i(-1)^l}{1536l} - \frac{143i(-1)^{1+l}}{10752l} - \frac{89i}{280(1+l)} - \frac{1391i(-1)^l}{17920(1+l)} - \frac{5i(-1)^{1+l}}{512(1+l)} \\
& - \frac{221i}{160(2+l)} - \frac{53i(-1)^l}{384(2+l)} + \frac{1043i(-1)^{1+l}}{1920(2+l)} - \frac{5i}{(3+l)^2} - \frac{5i(-1)^{1+l}}{(3+l)^2} + \frac{1009i}{240(3+l)} \\
& - \frac{12493i(-1)^l}{1920(3+l)} + \frac{213i(-1)^{1+l}}{128(3+l)} + \frac{15i}{2(4+l)^2} + \frac{15i(-1)^l}{2(4+l)^2} + \frac{105i}{32(4+l)} + \frac{279i(-1)^l}{128(4+l)} \\
& - \frac{141i(-1)^{1+l}}{128(4+l)} - \frac{5i}{(5+l)^2} - \frac{5i(-1)^{1+l}}{(5+l)^2} - \frac{257i}{24(5+l)} + \frac{187i(-1)^l}{384(5+l)} - \frac{263i(-1)^{1+l}}{128(5+l)} \\
& - \frac{217i}{8(6+l)^2} + \frac{7i(-1)^l}{8(6+l)^2} + \frac{31759i}{480(6+l)} + \frac{811i(-1)^l}{640(6+l)} - \frac{2683i(-1)^{1+l}}{1920(6+l)} - \frac{291i}{4(7+l)^2} \\
& - \frac{3i(-1)^{1+l}}{4(7+l)^2} - \frac{56809i}{560(7+l)} - \frac{2399i(-1)^l}{4480(7+l)} - \frac{2151i(-1)^{1+l}}{4480(7+l)} + \frac{545i}{8(8+l)^2} + \frac{i(-1)^l}{8(8+l)^2} \\
& + \frac{175459i}{3360(8+l)} + \frac{3429i(-1)^l}{17920(8+l)} - \frac{14977i(-1)^{1+l}}{53760(8+l)} - \frac{12i}{9+l} + \frac{i(-1)^l}{512(9+l)} + \frac{i(-1)^{1+l}}{512(9+l)}
\end{aligned}$$

Generic Spin

The coefficients multiplying 1/l! can be easily resummed to give log and Dilog functions and the generic structure reads where only the polynomial functions needs to be specified from the explicit computation.

$$\sum_l (l!)\, a_l\, \omega^l \sim \frac{1}{\omega^{2s+1}} \Big[p_1^{2s+1}(\omega) \log(1+\omega) + p_2^{2s+1}(\omega) \log(1-\omega) + p_3^{2s+1}(\omega) \mathrm{Li}_2(\omega) + p_4^{2s+1}(\omega) \mathrm{Li}_2(-\omega) \Big]$$

Eliminating the factorial is natural from the point of view of the following basis:

$$J \sim \oint d\tau\, f(\tau) e^{i\tau \xi^\alpha \eta_\alpha} J^{\text{can.}} \qquad \omega = \frac{1}{\tau}$$

The exact form is rather complicated

$$\mathbb{H}^0(D, CC) \neq \emptyset$$

$$\mathcal{V} \sim D(J + \mathbb{H}^0(D, CC))$$

$$J^{(0)}_{m,n} = \frac{\omega^2}{8(1-\omega^2)} \int_0^1 dt \Big[2\omega n \left(\frac{(1-t)^{m+2}(1+t)^n}{(m+1)(\omega t - 1)} + \frac{(t+1)^{m+2}(1-t)^n}{(m+1)(\omega t + 1)} \right)$$
$$- \frac{4\omega\left(1-t^2\right)(m+n+1)(1-t)^{m+n}}{(m+1)(\omega t + 1)}$$
$$+ \frac{t(2mn+m+3n+2)(1-t)^m(1+t)^n}{(m+1)(n+1)} + \frac{t(2mn+m+3n+2)(1+t)^m(1-t)^n}{(m+1)(n+1)}$$
$$+ \frac{(1-t)^m(1+t)^n(2(m+1)(n+1) - \omega(m+n+2))}{\omega(m+1)(n+1)}$$
$$+ \frac{(1+t)^m(1-t)^n(\omega(m+n+2) - 2(m+1)(n+1))}{\omega(m+1)(n+1)} + \frac{2(m-n)(1-t)^{m+n+1}}{(m+1)(n+1)} \Big]$$

$$J \sim \sum_l g_l \Box^l (\Phi^* \nabla^{n+m} \Phi + \dots)$$

$$J \sim g_1 \omega + g_2 \omega^2 + \dots$$

A convenient basis

The decomposition of the backreaction in the various currents and improvements is most easily obtained up to a choice of basis:

Prokushkin-Vasiliev 1999

$$C(y) = \int d^2\xi e^{iy\xi} \hat{C}(\xi)$$

$$J(y) \sim \int d^2\xi d^2\eta K(\xi, \eta, y) C(\xi) C(\eta)$$

Three possible tensor contraction can be defined:

$$\xi^\alpha \eta_\alpha \quad y^\alpha \eta_\alpha \quad y^\alpha \xi_\alpha$$

D is diagonal if we consider monomials of the type:

$$[y^\alpha(\xi+\eta)_\alpha]^n [y^\alpha(\xi-\eta)_\alpha]^m f(\xi^\alpha \eta_\alpha)$$

$$D\left([y^\alpha(\xi+\eta)_\alpha]^n [y^\alpha(\xi-\eta)_\alpha]^m f(\xi^\alpha \eta_\alpha)\right) = [y^\alpha(\xi+\eta)_\alpha]^n [y^\alpha(\xi-\eta)_\alpha]^m (\mathcal{D} f)(\xi^\alpha \eta_\alpha)$$

Cohomology Analysis

Non-trivial cohomologies cannot be removed by a (pseudo-)local redefinition

$$\mathcal{V} \neq D(\ldots)$$

$$\mathbb{H}^1(D, CC) \neq \emptyset$$

(Prokushkin-Vasiliev 1999)

$$D\widetilde{C}^{(2)} = \widetilde{\mathcal{V}}(\Omega, C^{(1)}, C^{(1)})$$

2s+1 cohomologies for each irreducible AdS irrep

$$\mathbb{H}^2(\widetilde{D}, \omega C) \neq \emptyset$$

$$\widetilde{D}\tilde{\omega}^{(2)} = \widetilde{\mathcal{V}}(\Omega, \omega^{(1)}, C^{(1)})$$

The backreaction of the scalar on the twisted sector might be irremovable

$$\mathbb{H}^2(\widetilde{D}, C) = \emptyset$$

but

$$\mathbb{H}^1(\widetilde{D}, C) \neq \emptyset$$

$$\widetilde{D}\tilde{\omega}^{(2)} = \widetilde{\mathcal{V}}(\Omega, \Omega, C^{(2)})$$

Further ambiguity is introduced to remove

$$\widetilde{\mathcal{V}}(\Omega, \Omega, C^{(2)}) \longrightarrow g_1\,,\, d_1$$

and shows up already in $\widetilde{\mathcal{V}}(\Omega, \omega^{(1)}, C^{(1)})$

Ordinary Lagrangian Formulation

We can now compare the induced gauge transformations at the action level with those appearing in Vasiliev's equations

$$\delta^{(s)}\Phi = 2ig_s\,\xi_{\underline{m}(s-1)}(2i\nabla^{\underline{m}})^{s-1}\Phi \quad \Longleftrightarrow \quad \delta C = [\xi_\omega, C] + \{\xi_e, C\}$$

The comparison allows fixing the constant g_s otherwise arbitrary at cubic order

$$g_s = \frac{1}{(2s-2)!}$$

A redefinition bringing from Vasiliev's backreaction to the above currents and coefficient is still pseudo-local but should be physically acceptable

Integration Flow

About the possibility of truncating away the twisted sector we already know a yes go result to all orders → Prokushkin-Vasiliev 1998

Consider a vacuum solution where: $\tilde{C}(y=0) = \nu$

$$\mathcal{B} = \nu + \mu\mathcal{B}'(\mu,\nu) \qquad \mathcal{W} = \mathcal{W}(\mu,\nu) \qquad S_\alpha = S_\alpha(\mu,\nu)$$

$$\begin{cases} \dfrac{\partial \mathcal{W}}{\partial \mu} = \dfrac{1}{2}\mathcal{B}' \star \dfrac{\partial \mathcal{W}}{\partial \nu} + \dfrac{1}{2}\dfrac{\partial \mathcal{W}}{\partial \nu} \star \mathcal{B}' \\ \dfrac{\partial \mathcal{B}'}{\partial \mu} = \dfrac{1}{2}\mathcal{B}' \star \dfrac{\partial \mathcal{B}'}{\partial \nu} + \dfrac{1}{2}\dfrac{\partial \mathcal{B}'}{\partial \nu} \star \mathcal{B}' \\ \dfrac{\partial \mathcal{S}_\alpha}{\partial \mu} = \dfrac{1}{2}\mathcal{B}' \star \dfrac{\partial \mathcal{S}_\alpha}{\partial \nu} + \dfrac{1}{2}\dfrac{\partial \mathcal{S}_\alpha}{\partial \nu} \star \mathcal{B}' \end{cases}$$

μ is the perturbative expansion parameter

Solution at μ=1 obtained from solution at μ=0

Compatible with the equations: It can be thought of as field redefinition mapping the non-linear system to the free one (also Fronsdal and Scalar)

Yes-go: twisted fields can be truncated away -- can we say something more?