Interactions in $4d$ Vasiliev theory

Higher Spin Theories and Holography

Zhenya Skvortsov

Albert Einstein Institute and Lebedev Institute

June, 4, 2015

*

*Based on the work to in progress, with N.Boulanger, P.Kessel and M.Taronna

Intro

Second-order equations or Cubic Lagrangian is the first nontrivial instance of HS interactions

Second-order equations, being derived from complete Vasiliev equations, always know more than is accessible via genuine cubic Noether procedure (Cubic action can be reconstructed by comparing with equations$\rightarrow$Massimo's talk).

Second order equations lead to many interesting puzzles that are crucial for understanding of AdS/CFT and HS (some correlators have been rigorously computed in Giombi-Yin, 2009; but ... infinities, puzzles for the rest).

$C(s_1, s_2; 0)$ and $C(0, s_1; s_2)$ for $s_1 > s_2$. The first problem is that $C(s, s; 0) = 0$. Another one is that $C(0, 0; s)$ is inconsistent.

It is remarkable that interaction vertices in Vasiliev theory are very close to CFT correlation functions — singularity at coincident boundary points, the behaviour is not present in local theory.

Moreover, one can show (V.Didenko, J.Mei, E.S, unpublished, also Colombo and Sundell) that in $W = 0$ gauge it is very easy to regularize kernels as to collapse B near the boundary to traces

$$tr(K \star .... \star K) = \langle j ... j \rangle$$

where $K \star \varkappa$ is a propagator for B. This explains and generalizes the Giombi-Yin result, but ...

Plan

We would like to systematically derive the interaction vertices from the $4d$ Vasiliev theory; to make contact with the usual methods of AdS/CFT, in particular, to work out the dictionary with the metric-like approach and to investigate the problems of extracting the correlation functions directly from the *unfolded equations*, which immediately leads to a problem of admissible functions vs. locality.

Unfolded HS theory

The unfolded equations that describe HS interactions read

$$d\omega = F^{\omega}(\omega, C) \qquad \omega\text{-HS connection}$$
$$dC = F^{C}(\omega, C) \qquad C\text{-Weyl tensors+der.}$$

where the expansion is in HS Weyl tensors C

$$F^{\omega}(\omega, C) = \omega \star \omega + \mathcal{V}(\omega, \omega, C) + \mathcal{V}(\omega, \omega, C, C) + ...$$
$$F^{C}(\omega, C) = \omega \star C - C \star \tilde{\omega} + \mathcal{V}(\omega, C, C) + ...$$

and F's are constrained by Frobenius integrability condition $d^2 \equiv 0$, which implies certain gauge symmetry.

Perturbative C-expansion is effectively re-summed by Vasiliev equations — need (?) to dig therein to extract F's

Higher-spin algebra in 4d

Isomorphism $so(3,2) \sim sp(4,\mathbb{R})$ allows (Vasiliev, 1986) to give a simple realization of the HS algebra. A quartet of canonical oscillators

$$[Y^A, Y^B] = 2iC^{AB} \qquad C^{AB} \text{ is an } sp(4) \text{ metric}$$

leads to oscillator realization of $sp(4)$ via bilinears

$$T^{AB} = -\frac{i}{4}\{Y^A, Y^B\} \qquad [T^{AB}, T^{CD}] = T^{AD}C^{BC} + \ldots$$

and then can be extended to all reasonable functions of Y

$$f(Y) \star g(Y) = \exp i\overleftarrow{\partial}_A C^{AB}\overrightarrow{\partial}_B = \int e^{iV^A U_A} f(Y+U)g(Y+V)$$

Sometimes it is convenient to split Y^A into y^α and $\bar{y}^{\dot{\alpha}}$

First order

Free equations read:

$$\mathcal{D}\omega = \mathcal{V}(h,h,C) \qquad \omega^{\alpha(s-1\pm k),\dot\alpha(s-1\mp k)}$$

$$\widetilde{\mathrm{D}}C = 0 \qquad C^{\alpha(2s+k),\dot\alpha(k)} \oplus C^{\alpha(k),\dot\alpha(2s+k)}$$

and are equivalent to Fronsdal equation

$$(\Box + m^2)\phi_{\underline{m}(s)} + ... = 0$$

imposed on the symmetric part of the HS vielbein

$$\phi_{\underline{m}(s)} = e^{\alpha(s-1),\dot\alpha(s-1)}_{\underline{m}} h_{\underline{m}|\alpha\dot\alpha}...h_{\underline{m}|\alpha\dot\alpha}$$

Cocycle $\mathcal{V}(h,h,C)$ defines the Weyl tensor

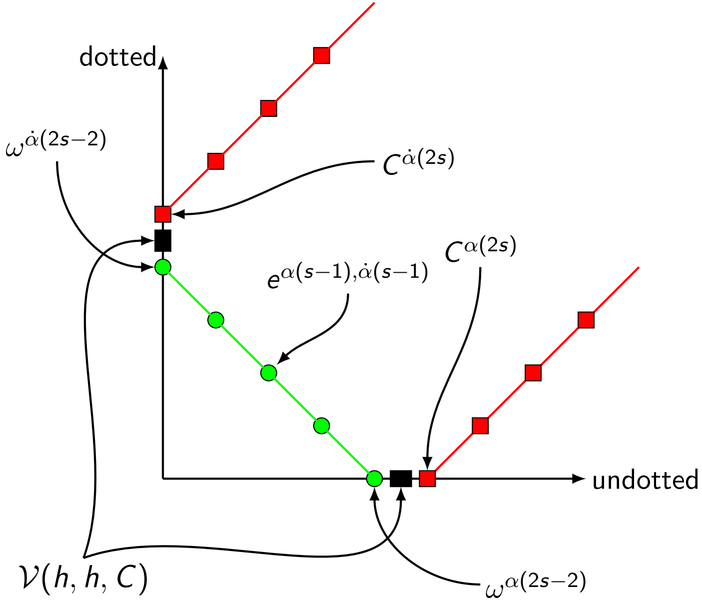
$$\nabla_{\underline{n}(s)}\phi_{\underline{m}(s)} = C_{\underline{n}(s),\underline{m}(s)}$$

which obeys the Bianchi identities

$$\nabla_{[\underline{r}}C_{\underline{n}(s-1)\underline{r},\underline{m}(s-1)\underline{r}]} = 0$$

The rest are derivatives of the Fronsdal field or pure gauge.

First order


dotted
$\omega^{\dot\alpha(2s-2)}$
$C^{\dot\alpha(2s)}$
$C^{\alpha(2s)}$
$e^{\alpha(s-1),\dot\alpha(s-1)}$
undotted
$\mathcal{V}(h,h,C)$
$\omega^{\alpha(2s-2)}$

Second order

The most general equations at the second order are

$$\mathcal{D}\omega_2 = \omega\star\omega + \mathcal{V}(h,\omega,C) + \mathcal{V}(h,h,C,C) + \mathcal{V}(h,h,C_2)$$
$$\widetilde{\mathrm{D}}C_2 = \omega\star C - C\star\tilde{\omega} + \mathcal{V}(h,C,C)$$

where some of the cocycles are explicitly determined by the HS algebra. $\omega\star C - C\star\tilde{\omega}$ is the only tested!

It is important to have nontrivial interactions. Purely star-product interactions should be (!) trivial

$$d\omega = \omega\star\omega + \omega\star\omega\star C + \ldots$$

as they can be redefined $\omega \to \omega + \omega\star C$. The difference between this and nontrivial is subtle.

Vasiliev equations

This is what we solve to the second order

$$
\begin{aligned}
&dW = W \star W\,, \\
&d(B\star\varkappa) = [W, B\star\varkappa]_\star\,, && d(B\star\bar{\varkappa}) = [W, B\star\bar{\varkappa}]_\star\,, \\
&dS_\alpha = [W, S_\alpha]_\star\,, && d\bar{S}_{\dot\alpha} = [W, \bar{S}_{\dot\alpha}]_\star\,, \\
&[S_\alpha, S_\beta]_\star = -2i\epsilon_{\alpha\beta}(1+e^{i\theta}B\star\varkappa)\,, && [\bar{S}_{\dot\alpha}, \bar{S}_{\dot\beta}]_\star = -2i\epsilon_{\dot\alpha\dot\beta}(1+e^{-i\theta}B\star\bar{\varkappa})\,, \\
&\{S_\alpha, B\star\varkappa\}_\star = 0\,, && \{\bar{S}_{\dot\alpha}, B\star\bar{\varkappa}\}_\star = 0\,, \\
&[S_\alpha, \bar{S}_{\dot\alpha}]_\star = 0\,,
\end{aligned}
$$

where $\varkappa = e^{iz_\alpha y^\alpha}$, $\bar{\varkappa} = e^{i\bar{z}_{\dot\alpha}\bar{y}^{\dot\alpha}}$ and all fields take values in an extension of the HS algebra with four additional Z^A oscillators that are crucial for the equations to lead to HS interactions.

Extracting unfolded equations

Shifting everything by the vacuum $W = AdS$, $S = Z_C dZ^C + A$

$$\begin{aligned} \partial A &= A \star A + B \star \Upsilon \\ \partial B &= A \star B - B \star \Upsilon^{-1} \star A \star \Upsilon \\ \partial W &= -[h + W, A] \end{aligned}$$

where $\partial = dZ^A \partial^Z_A$, $A = A_C dZ^C$ and $\Upsilon = (\varkappa e^{+i\theta}, \bar{\varkappa} e^{-i\theta})$, h is a vielbein. One can solve for the Z-dependence (in the Schwinger-Fock gauge $Z^C A_C = 0$)

$$\begin{aligned} A &= \partial^{-1}(A \star A + B \star \Upsilon) \\ B &= C(Y) + \partial^{-1}(A \star B - B \star \Upsilon^{-1} \star A \star \Upsilon) \\ W &= \omega(Y) - \partial^{-1}[h + W, A] \end{aligned}$$

where $C(Y)$, $\omega(Y)$ are the physical fields we are looking for equations for.

Extracting unfolded equations

The solutions for the Z-evolution need to be plugged into the two last equations

$$DW = W \star W + \text{Lorentz} \qquad \widetilde{\text{D}}B = W \star B - B \star \tilde{W}$$

to extract the equations in terms of $C(Y)$ and $\omega(Y)$. There is also an additional piece due to the requirement for the true Lorentz generators to preserve the Schwinger-Fock gauge, otherwise the spin-connection will appear outside the covariant derivative, which, for example, makes it difficult to relate HS vielbeins to Fronsdal fields. Lorentz redefinition contributes to the stress-tensors.

Second-order summary

we used Fourier transformed fields

$$C(y,\bar{y}|x)=\int d^4\xi\, e^{iY\xi}C(\xi|x)\,.$$

and for the most complicated cocycle $\mathcal{V}(h,h,C,C)$

$$\mathcal{V}(h,h,C,C)=\int d^2\xi\, d^2\eta\, H^{\alpha\alpha}J_{\alpha\alpha}(Y,\xi,\eta)\,C(\xi|x)C(\eta|x)+h.c.\,,$$

where $H^{\alpha\alpha}$ is a basis two-form (vielbein squared). Applying ∂^{-1} contributes homotopy integrals (at most two now)

$$\partial_\nu f^\nu = g(z) \qquad\qquad f_\alpha = z_\alpha \int_0^1 dt\, t\, g(zt)$$

Second-order summary

The most complicated cocycle $\mathcal{V}(h,h,C,C)$ still fits the slide

$$J = H^{\alpha\alpha}(y+\xi)_\alpha(y+\eta)_\alpha Q\left(iq^2t^2+(\bar{\xi}\bar{\eta})\frac{qt(1-qt)}{2}\right)+$$
$$-\frac{i}{2}H^{\dot{\alpha}\dot{\alpha}}\bar{\xi}_{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}}Q+$$
$$+\frac{i}{2}(1-t)H^{\dot{\alpha}\dot{\alpha}}\bar{\xi}_{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}}P+\frac{i}{2}H^{\dot{\alpha}\dot{\alpha}}\partial_{\dot{\alpha}}\partial_{\dot{\alpha}}K+h.c.$$

(the very first term is due to the Lorentz redefinition)

$$K = \exp i\left(t\eta\xi+(\bar{y}-\bar{\eta})(\bar{y}+\bar{\xi})+2\theta\right)$$
$$Q = \exp i\left((qt(y+\eta)(y+\xi)+(\bar{y}-\bar{\eta})(\bar{y}+\bar{\xi})+2\theta\right)$$
$$P = Q\Big|_{q=1}$$

This is to be compared with the $3d$ case!

Checking consistency

$$\mathcal{D}\omega_2 = \omega \star \omega + \mathcal{V}(h,\omega,C) + \mathcal{V}(h,h,C,C) + \mathcal{V}(h,h,C_2)$$
$$\widetilde{\mathrm{D}}C_2 = \omega \star C - C \star \tilde{\omega} + \mathcal{V}(h,C,C)$$

There are components on the r.h.s. of $D\omega_2 = ...$ that redefines Weyl tensors C_2. Therefore C_2 is not the Weyl tensor and its identification as order-s derivative of the Fronsdal tensor is wrong (but perhaps this still can be true near the boundary as in Giombi and Yin (puzzles))!

The cocyles are not D-closed independently, but cocyle $\mathcal{V}(h,h,C,C)$ (after K is dropped) is D-conserved:

$$D\mathcal{V}(h,h,C,C) = 0$$

which makes it possible to identify it as stress-tensors.

Frame-like vs. metric-like dictionary

In general at the second order one expects to find a chain of Fronsdal equations with sources

$$(\Box + m^2)\phi_{\underline{m}(s)} + ... = j_{\underline{m}(s)}(\phi, \phi)$$

Projecting onto the Weyl tensor simplifies equations a lot

$$(\Box + M^2)\nabla_{\underline{n}(s)}\phi_{\underline{m}(s)} = \nabla_{\underline{n}(s)} j_{\underline{m}(s)}(\phi, \phi)$$

for example, the canonical scalar field's stress-tensors consists of one term only

$$\phi \overleftrightarrow{\nabla}_{\underline{m}(s)} \phi + O(\Lambda) \qquad \Longrightarrow \qquad \nabla_{\underline{n}(s)}\phi \, \nabla_{\underline{m}(s)}\phi$$

Solving for Torsion

We need the first two equations

$$\nabla e + \sigma_- \omega_1 = J_0$$
$$\nabla \omega_1 + \sigma_- \omega_2 + \sigma_+ e = J_1$$

HS equations in unfolded form always have a non-vanishing Torsion (solving for it destroys the beautiful structure):

$$\omega_1 = \sigma_-^{-1} J_0 - \sigma_-^{-1} \nabla e$$

Plugging it to the second one and projecting redundant component we find

$$\sigma_+ e - \nabla \sigma_-^{-1} \nabla e = j \qquad\qquad j = J_1 - \nabla \sigma_-^{-1} J_0$$

The l.h.s. makes the Fronsdal operator

Getting Fronsdal current

Everything can be expanded as (canonical form)

$$\omega(y,\bar{y}) = h^{\alpha\alpha}\partial_\alpha\partial_{\dot\alpha}\omega^1 + y^\alpha h_\alpha{}^{\dot\alpha}\,\partial_{\dot\alpha}\omega^2 + y^{\dot\alpha} h^\alpha{}_{\dot\alpha}\,\partial_\alpha\omega^3 + y^\alpha y^{\dot\alpha} h_{\alpha\dot\alpha}\omega^4$$

which for any structure, e.g. J_0, costs no more than two homotopy integrals. Then, σ_-^{-1} has number operators N^{-1} or $(N+2)^{-1}$, which adds one more integral:

$$(N+2)^{-1} f(y) = \int_0^1 dt\, t\, f(ty)$$

Taking ∇ again destroys the canonical form, which costs one more integral

$$f_\alpha(y) = \partial_\alpha\left[N^{-1} y^\nu f_\nu\right] + y_\alpha\left[(N+2)^{-1}\partial_\nu f^\nu\right]$$

We end up with no more than 6 integrals in total, which can bring L^{-6} at best, where L is the number of contracted indices

Getting ∇^s of the Fronsdal current

$s-0-0$ is a closed subsector corresponding to $\mathcal{V}(h,C,C)$

$$\widetilde{\mathrm{D}}C_2=\omega\star C-C\star\tilde{\omega}+\mathcal{V}(h,C,C)$$

The problem is analogous to solving for Torsion (two first order equations with a source instead of a single second order equation for Weyl tensor $C^{\dot{\alpha}(2s)}$)

$$(\Box-(4+2\bar{N}))C(0,\bar{y})=\int 2(\bar{y}\bar{\xi}+\bar{y}\bar{\eta}))e^{i[t\eta\xi+(\bar{y}-\bar{\eta})(\bar{y}+\bar{\xi})+\theta]}C(\xi)C(\eta)+h.c.$$

In particular we can see that the source vanishes for the scalar $\bar{y}=0$ and therefore there is no scalar self-coupling as was observed by Sezgin and Sundell, which is in accordance with the $O(n)$-model.

Pseudo-local stress-tensors vs. improvements

In HS we generally find, e.g. the Fronsdal current,

$$j_{\underline{m}(s)} = \sum_k a_k \nabla_{\underline{m}} .. \nabla_{\underline{m}} \nabla_{\underline{n}(k)} \Phi \nabla_{\underline{m}} .. \nabla_{\underline{m}} \nabla^{\underline{n}(k)} \Phi$$

From flat (ambient) space we know that there are

- canonical currents $j^{can}_{a(s)} = \phi \overleftrightarrow{\partial}_{a(s)} \phi$
- canonical currents with cross-contractions, i.e. $(\partial_1 \cdot \partial_2)^L j^{can}_{a(s)}$
- on-shell trivial terms $\sim \Box \phi$
- other improvements evaluated on-shell

the 3rd we cannot see now, the $2nd$ generate pseudo-local tails, the 4th can be quotiented out.

Canonical current sector

While HS symmetry downgraded to a simple AdS background leaves some remnants in the form of improvements, we do not expect those to contribute to three-point functions. The explicit projection is

$$J = H^{\alpha\alpha}\xi^-_\alpha\xi^-_\alpha Q\left(iq^2t^2 + (\bar{\xi}\bar{\eta})\frac{qt(1-qt)}{2}\right) +$$
$$-\frac{i}{2}H^{\dot\alpha\dot\alpha}\bar{\xi}^+_{\dot\alpha}\bar{\xi}^+_{\dot\alpha}Q +$$
$$+\frac{i}{2}(1-t)H^{\dot\alpha\dot\alpha}\bar{\xi}^+_{\dot\alpha}\bar{\xi}^+_{\dot\alpha}P + h.c.$$

$$\xi^\pm_\alpha = \xi_\alpha \pm \eta_\alpha$$

Pseudo-local stress-tensors vs. locality

In HS we generally find, e.g. the Fronsdal current,

$$j_{\underline{m}(s)} = \sum_k a_k \nabla_{\underline{m}}..\nabla_{\underline{m}}\nabla_{\underline{n}(k)}\Phi\nabla_{\underline{m}}..\nabla_{\underline{m}}\nabla^{\underline{n}(k)}\Phi$$

For example,

$$C\star C \sim j_{\alpha(s),\dot{\alpha}(s)} \sim \sum_{L,\bar{L}} \frac{1}{L!\bar{L}!} C_{\alpha(s)\nu(L),\dot{\nu}(\bar{L})} C^{\nu(L),\dot{\nu}(\bar{L})}{}_{\dot{\alpha}(s)}$$
$$\sim \sum_k \frac{1}{L!L!} \nabla_{\underline{n}(L)} C_{\alpha(s)} \nabla^{\underline{n}(L)} C_{\dot{\alpha}(s)}$$

Contractions can be eaten by $\Box \sim L^2$ which makes the series divergent. Therefore, pure star-product redefinitions/expansions are nonlocal.

Pseudo-local stress-tensors vs. locality

In HS we generally find, e.g. the Fronsdal current,

$$j_{\underline{m}(s)} = \sum_k a_k \nabla_{\underline{m}}..\nabla_{\underline{m}} \nabla_{\underline{n}(k)} \Phi \nabla_{\underline{m}}..\nabla_{\underline{m}} \nabla^{\underline{n}(k)} \Phi$$

One can take them seriously and plug into the action $\int d^d x\, \phi j$. One contraction of indices can be eaten by $\Box$ and integrated by parts, yielding some prefactor c_k. If the sum

$$\sum_k c_k a_k$$

is convergent then the pseudo-local expression is actually local. Otherwise, usual field-theory recipes cannot be applied. See Massimo's talk

Conclusions

- $4d$ Vasiliev theory at the second order is much more concise than the $3d$ one. The vertices are explicitly found.
- The frame-like vs. Fronsdal dictionary is worked out and the stress-tensors are derived.
- We also revealed some subtleties in AdS/CFT computations