Bosonic Fradkin-Tseytlin equations unfolded

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Motivation

Fronsdal eq.$\rightarrow$Unfolded form.$\rightarrow$AdS HS algebra$\rightarrow$ Nonlin. AdS HS theory

Fradkin-Tseytlin eq.$\rightarrow$Unfolded form.$\rightarrow$Conf. HS alg.$\rightarrow$ Nonlin. conf. HST

Outlook

1. Fradkin - Tseytlin equations
2. Unfolded formalism (briefly)
3. Candidates to conformal higher spin algebra
4. Unfolded formulation of Fradkin - Tseytlin equations
5. Conclusions

Fradkin - Tseytlin equations

$$\underbrace{\partial^{\alpha}{}_{\dot\alpha}\cdots\partial^{\alpha}{}_{\dot\alpha}}_{s}\varphi^{\alpha(s)\,;\,\dot\alpha(s)} = C^{\alpha(2s)}\,,$$

$$\underbrace{\partial_{\alpha}{}^{\dot\alpha}\cdots\partial_{\alpha}{}^{\dot\alpha}}_{s}C^{\alpha(2s)} = 0\,,$$

complex conjugate pair of equations.

E. Fradkin, A. Tseytlin (1985)

Gauge transformations

$$\delta\varphi^{\alpha(s)\,;\,\dot\beta(s)} = \partial^{\alpha\dot\alpha}\varepsilon^{\alpha(s-1)\,;\,\dot\beta(s-1)}\,.$$

Symmetrization wrt indices denoted by the same letter imposed.

Unfolded formalism (briefly)

Background 1-form connection

Spin connection is zero.

Frame field $\mathrm{d}x^m \delta_m^n \sigma_n^{\alpha\dot{\beta}} = \xi^{\alpha\dot{\beta}}$.

Unfolded formalism (briefly)

Unfolded system

$$\Big(\mathrm{d} + \underbrace{\xi^{\alpha\dot\beta}\mathcal{P}^{\mathrm{gauge}}_{\alpha\dot\beta}}_{\sigma_-^{\mathrm{gauge}}}\Big)\omega = \underbrace{\xi^{\alpha\dot\beta}\xi^{\gamma\dot\delta}\mathcal{H}_{\alpha\gamma;\dot\beta\dot\delta}}_{\sigma} C + \text{comp. conj. term}\,,$$

$$\Big(\mathrm{d} + \underbrace{\xi^{\alpha\dot\beta}\mathcal{P}^{\mathrm{Weyl}}_{\alpha\dot\beta}}_{\sigma_-^{\mathrm{Weyl}}}\Big)C = 0\,, \qquad \text{complex conjugate equation.}$$

Gauge symmetry

$$\delta\omega = \Big(\mathrm{d} + \sigma_-^{\mathrm{gauge}}\Big)\epsilon\,,$$

Here
$\omega = \xi^{\alpha\dot\beta}\omega_{\alpha\dot\beta}$ — the set of 1-forms taking values in $su(2,2)$-module $\mathcal{M}^{\mathrm{gauge}}$.
ϵ — set of 0-forms taking values in $su(2,2)$-module $\mathcal{M}^{\mathrm{gauge}}$.
C — the set of 0-forms taking values in $su(2,2)$-module $\mathcal{M}^{\mathrm{Weyl}}$.
$\mathcal{P}^{\mathrm{gauge}}_{\alpha\dot\beta}$ and $\mathcal{P}^{\mathrm{Weyl}}_{\alpha\dot\beta}$ operators of translation repres. in $\mathcal{M}^{\mathrm{gauge}}$ and $\mathcal{M}^{\mathrm{Weyl}}$.
$\mathcal{H}_{\alpha\gamma;\dot\beta\dot\delta}$ — 'glues' Weyl sector to the gauge sector.

Unfolded formalism (briefly)

Dynamical content of unfolded system

Nilpotent operator $\hat{\sigma}_{-} = \sigma_{-}^{\mathrm{gauge}} + \sigma_{-}^{\mathrm{Weyl}} + \bar{\sigma}_{-}^{\mathrm{Weyl}} + \sigma + \bar{\sigma}$.

- $H^0_{\hat{\sigma}_-}$ — differential gauge parameters;
- $H^1_{\hat{\sigma}_-}$ — dynamical fields;
- $H^0_{\hat{\sigma}_-}$ — dynamical equations;

O.S., M. Vasiliev (2000)
M. Vasiliev (2008)

Candidates to conformal higher spin algebra

Oscillators

$$a^{\alpha}\,,\, b_{\beta}\,,\, \bar{a}^{\dot{\alpha}}\,,\, \bar{b}_{\dot{\beta}}\,,$$

$$[b_{\beta}, a^{\alpha}]_{*} = \delta^{\alpha}_{\beta}\,, \quad [\bar{b}_{\dot{\beta}}, \bar{a}^{\dot{\alpha}}]_{*} = \delta^{\dot{\alpha}}_{\dot{\beta}}\,.$$

Star product

$$f * g = f \exp(\overleftrightarrow{\Delta}) g\,,$$

$$\overleftrightarrow{\Delta} = \frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial b}\cdot\frac{\overrightarrow{\partial}}{\partial a} - \frac{\overleftarrow{\partial}}{\partial a}\cdot\frac{\overrightarrow{\partial}}{\partial b} + \frac{\overleftarrow{\partial}}{\partial \bar{b}}\cdot\frac{\overrightarrow{\partial}}{\partial \bar{a}} - \frac{\overleftarrow{\partial}}{\partial \bar{a}}\cdot\frac{\overrightarrow{\partial}}{\partial \bar{b}}\right)\,,$$

where $a\cdot b = a^{\gamma} b_{\gamma}\,, \quad \bar{a}\cdot\bar{b} = \bar{a}^{\gamma}\bar{b}_{\dot{\gamma}}\,, \quad \frac{\partial}{\partial a}\cdot\frac{\partial}{\partial b} = \frac{\partial}{\partial a^{\gamma}}\cdot\frac{\partial}{\partial b_{\gamma}}$, etc.

Candidates to conformal higher spin algebra

Oscillator realization of $so(4,2) \sim su(2,2)$

1. Bilinear combinations of oscillators.
2. Centralized by $\mathcal{Z} = i/2(a \cdot b - \bar{a} \cdot \bar{b})$, i.e.

$$[\mathcal{Z}, f]_* = i/2(n_a - n_{\bar{a}} - n_b + n_{\bar{b}})f = 0\,,$$

   where $n_a\,,\; n_b\,,\; n_{\bar{a}}\,,\; n_{\bar{b}}$ are Euler operators.
3. Reality condition

$$\zeta(f) = f\,,$$

   where $\zeta(a^\alpha) = \bar{a}^{\dot{\alpha}}\,, \quad \zeta(b_\beta) = \bar{b}_{\dot{\beta}}$.
4. Factorized by $\mathcal{Z}$.

M. Günaydin (1983)

Infinite-dimensional extension $isu^\infty(2,2)$

Polylinear combinations of oscillators satisfying to items 2, 3. Factorized by all star powers of $\mathcal{Z}$. E. Fradkin, V. Linetsky (1990)

Candidates to conformal higher spin algebra

Infinite chain of ideals of $isu^{\infty}(2,2)$

$$isu^{\infty}(2,2) \supset \mathfrak{I}^1 \supset \mathfrak{I}^2 \supset \cdots \supset \mathfrak{I}^m \supset \cdots ,$$

where $\mathfrak{I}^m$ is spanned by elements $(\mathcal{Z}*)^m * f$.

Algebra $isu^m(2,2)$

$$isu^m(2,2) = isu^{\infty}(2,2)/\mathfrak{I}^{m+1} , \qquad m = 0, 1, \ldots, \infty .$$

System under consideration

Gauge module $\mathcal{M}^{\mathrm{gauge}}$

Let gauge $su(2,2)$-module $\mathcal{M}^{\mathrm{gauge}}$ be $\mathcal{M}^{\infty} = \mathrm{ad}_{su(2,2)}\Big(isu^{\infty}(2,2)\Big)$.

$$\mathcal{P}^{\infty}_{\alpha\dot{\beta}} = b_{\alpha}\frac{\partial}{\partial \bar{a}^{\dot{\beta}}} + \bar{b}_{\dot{\beta}}\frac{\partial}{\partial a^{\alpha}} \,.$$

Weyl module $\mathcal{M}^{\mathrm{Weyl}}$

Let Weyl $su(2,2)$-module $\mathcal{M}^{\mathrm{Weyl}}$ be $\tilde{\mathcal{M}}^{\infty}$ — the twist-transformed $\mathcal{M}^{\infty}$.

Twist transformation

$$b_{\alpha} \to \frac{\partial}{\partial \tilde{b}^{\alpha}}\,, \qquad \frac{\partial}{\partial b_{\alpha}} \to -\tilde{b}^{\alpha}\,,$$

$$[\frac{\partial}{\partial b_{\alpha}}, b_{\beta}] = [-\tilde{b}^{\alpha}, \frac{\partial}{\partial \tilde{b}^{\beta}}] = \delta^{\alpha}_{\beta}\,.$$

$$\tilde{\mathcal{P}}^{\infty}_{\alpha\dot{\beta}} = \frac{\partial^2}{\partial \tilde{b}^{\alpha}\partial \bar{a}^{\dot{\beta}}} + \bar{b}_{\dot{\beta}}\frac{\partial}{\partial a^{\alpha}} \,.$$

Unfolded system under consideration

$$\Big(\mathrm{d}+\xi^{\alpha\dot\beta}(b_\alpha\frac{\partial}{\partial\bar a^{\dot\beta}}+\bar b_{\dot\beta}\frac{\partial}{\partial a^\alpha})\Big)\omega^\infty=\Xi^{\dot\alpha\dot\alpha}\frac{\partial^2}{\partial\bar a^{\dot\alpha}\partial\bar a^{\dot\alpha}}\Big|_{\tilde b=0}C^\infty+\text{comp. conj. term}\,,$$

$$\Big(\mathrm{d}+\xi^{\alpha\dot\beta}(\frac{\partial^2}{\partial\tilde b^\alpha\partial\bar a^{\dot\beta}}+\bar b_{\dot\beta}\frac{\partial}{\partial a^\alpha})\Big)C^\infty=0\,,$$

complex conjugate equation for $\bar C^\infty$,

$$\delta\omega^\infty=\Big(\mathrm{d}+\xi^{\alpha\dot\beta}(b_\alpha\frac{\partial}{\partial\bar a^{\dot\beta}}+\bar b_{\dot\beta}\frac{\partial}{\partial a^\alpha})\Big)\epsilon^\infty\,,$$

where 2-form $\Xi^{\dot\alpha\dot\alpha}=\xi^{\beta\dot\alpha}\xi^{\gamma\dot\alpha}\epsilon_{\beta\gamma}$.

M. Vasiliev (2001)

Structure of module $\mathcal{M}^\infty$

1. Module $\mathcal{M}^\infty$ is reducible with submodules $\mathfrak{I}^m$ — the ideals of $isu^\infty(2,2)$. Let $\mathcal{M}^m = \mathcal{M}^\infty / \mathfrak{I}^{m+1}$.
2. Operator of spin

$$\hat{s} = n_a + n_{\bar{b}} + 1 = n_b + n_{\bar{a}} + 1$$

   commutes with $su(2,2)$. Let $\hat{s}\mathcal{M}^m_s = s\mathcal{M}^m_s$.
3. Module $\mathcal{M}^0_s$ is irreducible. Let denote it as $\mathcal{M}_s$.
4. The following decomposition is true

$$\mathcal{M}^m = (m+1) \oplus_{s=1}^{\infty} \mathcal{M}_s \,, \quad m = 0, 1, \ldots, \infty \,.$$

5. Elements of submodule $\mathcal{M}_s$ enter to the whole module $\mathcal{M}^m$ with the factor $\mathcal{Z}^q$, $q = 0, \ldots, m$.

Structure of module $\tilde{\mathcal{M}}^\infty$

Module $\tilde{\mathcal{M}}^\infty$ has analogous structure

$$\tilde{\mathcal{M}}^m = (m+1) \oplus_{s=1}^{\infty} \tilde{\mathcal{M}}_s \,, \quad m = 0, 1, \ldots, \infty \,,$$

where elements of submodule $\tilde{\mathcal{M}}_s$ enter to the whole module $\tilde{\mathcal{M}}^m$ with the factor $\tilde{\mathcal{Z}}^q$, $q = 0, \ldots, m$.

$$\tilde{\mathcal{Z}} = \frac{i}{2}(a \cdot \frac{\partial}{\partial \tilde{b}} - \bar{a} \cdot \bar{b}) \,.$$

Decomposition of unfolded system

The question is: if the unfolded system admits decomposition that modules do?

$$\Big(\mathrm{d}+\sigma_-\Big)\mathcal{Z}\omega_s = \sigma\tilde{\mathcal{Z}}C_s + \text{comp. conj. term}\,,$$
$$\Big(\mathrm{d}+\tilde{\sigma}_-\Big)\tilde{\mathcal{Z}}C_s = 0\,,$$
$$\text{comp. conj. equation for } \bar{C}_s.$$

Since $[\mathrm{d}+\sigma_-,\mathcal{Z}]=0$ and $[\mathrm{d}+\tilde{\sigma}_-,\tilde{\mathcal{Z}}]=0$ we can drag out $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ on the left had sides of equations.
We need $\sigma\tilde{\mathcal{Z}}=\mathcal{Z}\sigma$.

$$\sigma\tilde{\mathcal{Z}} = \mathcal{Z}\sigma + \sigma_-\psi + \psi\tilde{\sigma}_-\,,$$

where

$$\psi = i\xi^{\alpha\dot{\beta}}\epsilon_{\alpha\gamma}a^{\gamma}\frac{\partial}{\partial\bar{a}^{\dot{\beta}}}\Big|_{\tilde{b}=0}\,.$$

Field redefinition

$$\omega_s \to \omega_s - \psi C_s - \bar{\psi}\bar{C}_s\,.$$

$\hat{\sigma}_-$ cohomology

Gauge sector

$$H^0_s = \epsilon^{\beta(s-1);\dot\beta(s-1)} b_{\beta(s-1)} \bar b_{\dot\beta(s-1)}$$

$$H^1_s = \xi^{\gamma\dot\delta} \phi_{\gamma;\dot\delta}^{\beta(s-1);\dot\beta(s-1)} b_{\beta(s-1)} \bar b_{\dot\beta(s-1)}$$

$$H^2_s = \Xi^{\gamma\gamma} E^{\beta(s-1)}_{\alpha(s-1)\gamma(2)} a^{\alpha(s-1)} b_{\beta(s-1)} + \text{c.c.}$$

Weyl sector

$$\tilde H^0_s = C^{\dot\beta(s-1)}_{\dot\alpha(s+1)} \bar a^{\dot\alpha(s+1)} \bar b_{\dot\beta(s-1)}$$

$$\tilde H^1_s = \xi^{\gamma\dot\delta} \tilde E_{\beta(s-1)\gamma;\dot\alpha(s+1),\dot\delta} \tilde b^{\beta(s-1)} \bar a^{\dot\alpha(s+1)}$$

$$\left(\mathrm{d} + \sigma_-\right)\omega_s = 0 \longleftarrow \sigma C_s + \text{comp. conj. term}\,,$$

$$\left(\mathrm{d} + \tilde\sigma_-\right) C_s = 0\,,$$

comp. conj. equation for $\bar C_s$.

$$\underbrace{\partial^\alpha{}_{\dot\alpha} \cdots \partial^\alpha{}_{\dot\alpha}}_{s} \varphi^{\alpha(s)\,;\,\dot\alpha(s)} = 0 \longleftarrow C^{\alpha(2s)}\,,$$

$$\underbrace{\partial_\alpha{}^{\dot\alpha} \cdots \partial_\alpha{}^{\dot\alpha}}_{s} C^{\alpha(2s)} = 0\,,$$

and complex conjugate.

Conclusion

- The structure of unfolded system corresponding to algebra $isu^m(2,2)$ was explored.
- Specific field redefinition (mixing fields from gage sector and Weyl sector) that decomposes above system into subsystems corresponding to particular spin was found.
- It was shown that unfolded system corresponding to $isu^m(2,2)$ describes dynamical fields of all spins where each spin occurs $m+1$ times.