Aspects of 4D higher-spin gravity from exact solutions

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Summary

- The 4D Vasiliev equations
  - Oscillator algebras
  - Full equations (bosonic)
- Solving the equations
  - Gauge function method
  - Building blocks of exact solutions
- Exact solutions
  - Local data of exact solutions: continuous and discrete “moduli”
  - Zoology of known exact solutions
- Conclusions and Outlook

Oscillator algebra

- Commuting variables $Y_{\underline{\alpha}} = (y_\alpha, \bar{y}_{\dot\alpha})$, $Z_{\underline{\alpha}} = (z_\alpha, -\bar{z}_{\dot\alpha}) \to \mathfrak{sp}(4,\mathbb{R})$ quartets

$$[Y_{\underline{\alpha}}, Y_{\underline{\beta}}]_\star = 2iC_{\underline{\alpha\beta}} = 2i \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot\alpha\dot\beta} \end{pmatrix} , \qquad [Z_{\underline{\alpha}}, Z_{\underline{\beta}}]_\star = -2iC_{\underline{\alpha\beta}} , \quad [Y_{\underline{\alpha}}, Z_{\underline{\beta}}]_\star = 0$$

- Star-product:

$$\widehat{F}(Y,Z) \star \widehat{G}(Y,Z) = \int_{\mathcal{R}} \frac{d^4U d^4V}{(2\pi)^4} e^{iV^{\underline{\alpha}}U_{\underline{\alpha}}} \widehat{F}(Y+U, Z+U)\, \widehat{G}(Y+V, Z-V)$$

- π automorphism generated by the inner kleinian operator κ:

$$\pi(\widehat{f}(y,\bar{y};z,\bar{z})) = \widehat{f}(-y,\bar{y};-z,\bar{z}) , \quad \bar\pi(\widehat{f}(y,\bar{y};z,\bar{z})) = \widehat{f}(y,-\bar{y};z,-\bar{z})$$

$$\begin{aligned} \pi(\widehat{f}) &= \kappa \star \widehat{f} \star \kappa , \qquad \kappa = e^{iy^\alpha z_\alpha} , \qquad \kappa\star\kappa = 1 \\ \kappa &= \kappa_y \star \kappa_z , \quad \kappa_y \star \kappa_y = 1 \quad \textit{idem}\ \kappa_z,\ \bar\kappa_{\bar y} \text{ and } \bar\kappa_{\bar z} \\ \kappa_y &= 2\pi\delta^2(y) = 2\pi\delta(y_1)\delta(y_2) \end{aligned}$$

- Fields live on *correspondence space*, locally $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$:

$$d \to \hat{d} = d + d_Z = dx^\mu \frac{\partial}{\partial x^\mu} + dz^\alpha \frac{\partial}{\partial z^\alpha} + d\bar{z}^{\dot\alpha} \frac{\partial}{\partial \bar{z}^{\dot\alpha}}$$

$$A(x|Y) \to \hat{A}(x|Z,Y) \equiv (dx^\mu \hat{A}_\mu + dz^\alpha \hat{V}_\alpha + d\bar{z}^{\dot\alpha}\hat{V}_{\dot\alpha})(x|Z,Y) , \quad A_\mu(x|Y) = \hat{A}_\mu\big|_{Z=0}$$

$$\Phi(x|Y) \to \hat\Phi(x|Z,Y) , \quad \Phi(x|Y) = \hat\Phi(x|Z,Y)\big|_{Z=0}$$

The Vasiliev Equations

- Gauge field $\in$ Adj($\mathfrak{hs}(3,2)$) (*master 1-form connection*):

$$A_\mu(x|y,\bar{y}) = \sum_{n+m=2\mathrm{mod}4}^{\infty} \frac{i}{2n!m!} dx^\mu A_\mu{}^{\alpha_1\ldots\alpha_n\dot{\alpha}_1\ldots\dot{\alpha}_m}(x) y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot{\alpha}_1}\ldots\bar{y}_{\dot{\alpha}_m}$$

(every spin-s sector contains all one-form connections that are necessary for a frame-like formulation of HS dynamics (finitely many))

Generators of $\mathfrak{hs}(3,2)$: $T_s \sim y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot{\alpha}_1}\ldots\bar{y}_{\dot{\alpha}_m}\ ,\ \frac{n+m}{2}+1=s$

Bilinears in osc. → $\mathfrak{so}(3,2)$: $M_{AB} = -\frac{1}{8} Y^{\underline{\alpha}}(\Gamma_{AB})_{\underline{\alpha\beta}} Y^{\underline{\beta}} = \{M_{ab}, P_a\}$

- "Twisted adjoint" *0-form* (contains scalar, Weyl, HS Weyl and derivatives)

$$T(X)(\Phi) = [X,\Phi]_{\star,\pi} \equiv X\star\Phi - \Phi\star\pi(X)$$

- *Weyl 0-form* : $\Phi(x|y,\bar{y}) = \sum_{|n-m|=0\mathrm{mod}4}^{\infty} \frac{1}{n!m!} \Phi^{\alpha_1\ldots\alpha_n\dot{\alpha}_1\ldots\dot{\alpha}_m}(x)\, y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot{\alpha}_1}\ldots\bar{y}_{\dot{\alpha}_m}$

N.B.: spin-s sector → infinite-dimensional
(upon constraints, all on-shell-nontrivial covariant derivatives of the physical fields,
i.e., all the local dof encoded in the 0-form at a point)

The Vasiliev Equations

- Full eqs:

(Vasiliev '90)

$$\hat{F} \equiv \hat{d}\hat{A} + \hat{A} \star \hat{A} = \frac{i}{4}(dz^{\alpha} \wedge dz_{\alpha}\, \hat{\mathcal{B}} \star \hat{\Phi} \star \kappa + d\bar{z}^{\dot{\alpha}} \wedge d\bar{z}_{\dot{\alpha}} \hat{\bar{\mathcal{B}}} \star \hat{\Phi} \star \bar{\kappa})$$

$$\hat{\mathcal{D}}\hat{\Phi} \equiv \hat{d}\hat{\Phi} + \hat{A} \star \hat{\Phi} - \hat{\Phi} \star \bar{\pi}(\hat{A}) = 0$$

Local sym: $\delta\hat{A} = \hat{D}\hat{\epsilon}$, $\delta\hat{\Phi} = -[\hat{\epsilon}, \hat{\Phi}]_{\pi}$

- In components:

$$\widehat{F}_{\mu\nu} = \widehat{F}_{\mu\alpha} = \widehat{F}_{\mu\dot{\alpha}} = 0 , \qquad \widehat{D}_{\mu}\widehat{\Phi} = 0 ,$$

$$\left[\widehat{S}_{\alpha}, \widehat{S}_{\beta}\right]_{\star} = -2i\epsilon_{\alpha\beta}(1 - \mathcal{B} \star \widehat{\Phi} \star \kappa) ,$$

$$\left[\widehat{S}_{\dot{\alpha}}, \widehat{S}_{\dot{\beta}}\right]_{\star} = -2i\epsilon_{\dot{\alpha}\dot{\beta}}(1 - \bar{\mathcal{B}} \star \widehat{\Phi} \star \bar{\kappa})$$

$$\left[\widehat{S}_{\alpha}, \widehat{S}_{\dot{\beta}}\right]_{\star} = 0 ,$$

$$\widehat{S}_{\alpha} \star \widehat{\Phi} + \widehat{\Phi} \star \pi(\widehat{S}_{\alpha}) = 0 ,$$

$$\widehat{S}_{\dot{\alpha}} \star \widehat{\Phi} + \widehat{\Phi} \star \bar{\pi}(\widehat{S}_{\dot{\alpha}}) = 0$$

$$\hat{S}_{\alpha} = z_{\alpha} - 2i\hat{V}_{\alpha}$$

[Evolution along Z determines Z-contractions in terms of original dof.
Solution of Z-eqs. yields consistent nonlinear corrections as an expansion in Φ]

Exact solutions in HSGRA

- Crucial to look into the non-perturbative sector of the theory, may shed some light on peculiarities of HS physics and prompts to study global issues in HS gravity (boundary conditions, asymptotic charges, global dof in $\mathcal{Z}\ldots$).

- The Z-extended unfolded system encodes physical field equations that are highly non-local: at any order in the coupling constant → infinite derivative expansion.
  → radical departure from the familiar setups of lower-spin field theories, quantum effective theories or even SUGRA + stringy higher-derivative corrections.

- How to physically interpret the solutions? A few gauge-invariant observables are known, but their physical meaning is not always clear.
  Presently lacking a complete understanding, we shall classify some solutions according to the deformation parameters that activate them. Some of them are manifestly related to observables, for some other it is hard to say.

Exact solutions: gauge function method

- $\mathcal{X}$ x $\mathcal{Y}$ x $\mathcal{Z}$-space eqns:
- $\mathcal{Y}$ x $\mathcal{Z}$-space eqns:

$$
\begin{aligned}
\widehat{F}_{\mu\nu} &= \widehat{F}_{\mu\alpha} = \widehat{F}_{\mu\dot{\alpha}} = 0 \ , \qquad \widehat{D}_\mu \widehat{\Phi} = 0 \ , \\
\left[\widehat{S}'_\alpha, \widehat{S}'_\beta\right]_\star &= -2i\epsilon_{\alpha\beta}(1 - \mathcal{B} \star \widehat{\Phi}' \star \kappa) \ , \\
\left[\widehat{S}'_{\dot\alpha}, \widehat{S}'_{\dot\beta}\right]_\star &= -2i\epsilon_{\dot\alpha\dot\beta}(1 - \bar{\mathcal{B}} \star \widehat{\Phi}' \star \bar\kappa) \\
\left[\widehat{S}'_\alpha, \widehat{S}'_{\dot\beta}\right]_\star &= 0 \ , \\
\widehat{S}'_\alpha \star \widehat{\Phi}' + \widehat{\Phi}' \star \pi(\widehat{S}'_\alpha) &= 0 \ , \\
\widehat{S}'_{\dot\alpha} \star \widehat{\Phi}' + \widehat{\Phi}' \star \bar\pi(\widehat{S}'_{\dot\alpha}) &= 0
\end{aligned}
$$

- Can solve locally all equations with at least one spacetime component via some gauge function:

$$\hat{A}_\mu = \hat{L}^{-1} \star \partial_\mu \hat{L} \ , \quad \hat{S}_\alpha = \hat{L}^{-1} \star (\hat{S}'_\alpha) \star \hat{L} \ , \quad \hat{\Phi} = \hat{L}^{-1} \star \hat{\Phi}' \star \pi(\hat{L})$$

$$\hat{L} = \hat{L}(x|Z,Y) \ , \ \hat{L}(0|Z,Y) = 1 \qquad \hat{S}'_\alpha = \hat{S}_\alpha(0|Z,Y) \ , \quad \hat{\Phi}' = \hat{\Phi}(0|Z,Y)$$

- The remaining equations can be solved by various methods.
  Then “dress” all fields with x-dependence by performing star-products with the gauge function.

Gauge fields sector

- We want to interpret the coefficients of the master fields as space-time tensors → it should be possible to extract Lorentz tensors (and a Lorentz connection) out of the gauge fields generating function.

- But, in general, expansion coefficients in $\hat{A}_\mu$ are not Lorentz tensors!

- The proper Lorentz generator, at the full level, is

$$\widehat{M}_{\alpha\beta} = y_\alpha y_\beta - z_\alpha z_\beta + \frac{1}{2}\{\widehat{S}_\alpha, \widehat{S}_\beta\}_\star =: \widehat{M}^{(0)}_{\alpha\beta} + \widehat{M}^{(S)}_{\alpha\beta}$$

under which

$$\begin{aligned}
\delta_L \hat{\Phi} &\equiv -[\hat{\epsilon}_L, \hat{\Phi}]_\pi = -[\hat{\epsilon}_0, \hat{\Phi}]_\star \,, \\
\delta_L \widehat{A}_\alpha &\equiv \widehat{D}_\alpha \hat{\epsilon}_L = -[\hat{\epsilon}_0, \widehat{A}_\alpha]_* + \Lambda_\alpha{}^\beta \widehat{A}_\beta \,, \\
\delta_L \widehat{A}_\mu &\equiv \widehat{D}_\mu \hat{\epsilon}_L = -[\hat{\epsilon}_0, \widehat{A}_\mu]_\star + \left(\frac{1}{4i}\partial_\mu \Lambda^{\alpha\beta} \widehat{M}_{\alpha\beta} - \text{h.c.}\right) .
\end{aligned}$$

- Complicated, field-dependent transformation, but the field-redefinition

$$\widehat{A}_\mu - \frac{1}{4i}\omega_\mu^{\alpha\beta}\widehat{M}_{\alpha\beta} - \text{h.c.} = \widehat{W}_\mu$$

only contain Lorentz tensors!

"Moduli" space

- Solutions constructed assembling data from $\hat{\Phi}$', $\hat{S}$', $\hat{L}$. In general one should also consider transition functions T_I^J gluing together locally-defined field configurations.

$$\widehat{\Phi}_I = (\widehat{T}_I^{I'})^{-1} \star \widehat{\Phi}_{I'} \star \pi(\widehat{T}_I^{I'}) \,, \qquad \widehat{A}_I = (\widehat{T}_I^{I'})^{-1} \star (\widehat{A}_{I'} + \widehat{d}) \star \widehat{T}_I^{I'}$$

- What are the quantities that build the space of solutions?

1. Local dof in Φ'(Y) := $\hat{\Phi}$'(Y, Z)$|_{Z=0}$.

   Initial condition of Z-evolution. All on-shell nontrivial derivatives of physical fields at a spacetime point. The gauge function spreads this datum over space-time (more precisely, over a chart).

The functional form of Φ'(Y) matters, since, for any chosen gauge function, it contributes to the space-time behaviour of the fields, different asymptotics, etc. .
Moreover (and not disconnectedly) specific functions Φ'(Y) may have a peculiar behaviour under ★-product (span some subalgebra, diverge, etc.) → different SECTORS of HSGRA.

"Moduli" space

2. Monodromies and projectors in $\hat{V}'^{(0)}_{\alpha} = \hat{V}'_{\alpha}|_{\Phi'=0}$.

   It is particularly interesting that Z-space connection can be flat but nontrivial. New vacua? Global dof in Z?

3. Choice of gauge function $\hat{L}$: boundary dof may be contained in $\hat{L}\,|_{\partial C}$ (boundary values in (x,Y,Z) may affect observables)

4. Windings in transition functions T_I^J gluing together locally-defined field configurations

- N.B.: *AdS_4 vacuum solution* in the gauge function approach:

$$\widehat{\Phi} = 0\,, \quad \widehat{S}_\alpha = \widehat{S}^{(0)}_\alpha = z_\alpha\,, \quad \widehat{S}_{\dot\alpha} = \widehat{S}^{(0)}_{\dot\alpha} = \bar{z}_{\dot\alpha}\,, \quad \widehat{A}_\mu = \Omega^{(0)}_\mu = L^{-1}\star\partial_\mu L$$

$$L(x;y,\bar{y}) = e_\star^{i\lambda\tilde{x}^\mu(x)\delta^a_\mu P_a} \quad : \quad \mathcal{R}^{3,1} \longrightarrow \frac{SO(3,2)}{SO(3,1)} \quad \longrightarrow \quad ds^2_{(0)} = \frac{4dx^2}{(1-x^2)^2}$$

I. Weyl 0-form moduli and "sectors" of HSGRA

- Weyl 0-form moduli have so far been explored in the following sectors:

  a. <u>TWISTOR SPACE POLYNOMIALS AND PLANE WAVES</u> ($\widehat{\Phi}' = e^{i\lambda^\alpha y_\alpha + i\bar{\lambda}^{\dot\alpha}\bar{y}_{\dot\alpha}}$)

  The only Weyl 0-form initial datum dressed into a full solution is the trivial polynomial $\widehat{\Phi}' = \nu \in \mathbf{R}$ → $\mathfrak{so}(3,1)$-invariant solution

$$\widehat{\Phi}' = \nu\ , \qquad \widehat{S}'_\alpha = z_\alpha S(y^\alpha z_\alpha)\ , \qquad \widehat{S}_{\dot\alpha} = \bar{z}_{\dot\alpha}\bar{S}(\bar{y}^{\dot\alpha}\bar{z}_{\dot\alpha})$$

  → gives rise to a scalar profile on a rescaled AdS metric

$$\phi(x) = \nu(1-x^2)\ , \qquad ds^2 = \frac{4\Omega^2 d\tilde{x}^2}{(1-\tilde{x}^2)^2}$$

(Sezgin-Sundell '05)

  [Possible interesting dual interpretation, but complicated analysis at finite *v* *(C.I., J. Raeymaekers, in progress)*]

I. Weyl 0-form moduli and "sectors" of HSGRA

b. <u>States with compact $\mathfrak{so}(3,2)$-weights</u>

Φ' = enveloping-algebra realization of states with definite eigenvalues of E and spatial rotations M_{rs}, that are organized in Harish-Chandra modules $\mathcal{M}$ under the action of $\mathfrak{so}(3,2)$.

$$\widehat{\Phi}'(Y,Z) \ = \ \Phi'(Y) \ \in \ \mathcal{M} \ = \ \bigoplus_{e,s} \mathbf{C} \otimes T_{e,s}$$

It can be shown that they form indecomposable modules, comprising the AdS massless (anti-)particle (highest-) lowest-weight modules $\mathcal{D}(e_0, s_0)$ PLUS a wedge $\mathcal{W}$ of states of intermediate E-eigenvalues (corresponding to runaway modes of the linearized theory). More precisely,

$$\boxed{\mathcal{M} \ = \ \mathcal{W} \supset\!\!\!\!+ \mathcal{D}}$$

The $\mathfrak{so}(3,2)$ action can take from $\mathcal{W}$ to the particle subspace $\mathcal{D}$, but not back out → the entire $\mathcal{M}$ can be generated from some reference state(s) in $\mathcal{W}$.

I. Weyl 0-form moduli and "sectors" of HSGRA

- For example, the non-polynomial element $\mathcal{P}_1 := 4\, e^{-4E}$ has the properties:

$$\begin{aligned} \mathcal{P}_1 \star \mathcal{P}_1 &= \mathcal{P}_1 \\ E \star \mathcal{P}_1 &= \mathcal{P}_1 \star E = \frac{1}{2}\mathcal{P}_1 , \\ L_r^- \star \mathcal{P}_1 &= 0 = \mathcal{P}_1 \star L_r^+ , \\ M_{rs} \star \mathcal{P}_1 &= 0 \end{aligned} \quad \Rightarrow$$

$$\boxed{\mathcal{P}_1(E) \simeq |1/2;0\rangle\langle 1/2;0| \in \mathcal{D}_0 \otimes \mathcal{D}_0^*}$$

and from the point-of-view of the two-sided, twisted-adjoint action $K \star \mathcal{P}_1 - \mathcal{P}_1 \star \pi(K)$

$$\boxed{\mathcal{P}_1(E) \simeq |1;0\rangle \in \mathcal{D}(1,0)}$$

(C.I., P. Sundell '08)

- The $\mathfrak{so}(3,2)$ action on $\mathcal{P}_1$ reconstructs the projectors on all other $\mathcal{D}(1,0)$ modes → the $\mathfrak{hs}(3,2)$ action gives the modes of all spin-s AdS massless particles.

- Interestingly, one can generate all (anti-)particle modules via twisted adjoint $\mathfrak{hs}(3,2)$-action from the *static* runaway mode(s) $\phi_{0;(0)} = \sinh(4E)/4E$ (and $\phi_{0;(1)}$) of the free scalar field .

- It is unclear whether these static elements can be dressed up to full solutions [partly because they are not singleton composites].

I. Weyl 0-form moduli and "sectors" of HSGRA

- However, there exists another sector containing static states that, by construction, have much simpler non-linear completion.
- Its elements can be obtained from particle states by ★-multiplication with κ_y , forming the space $S = \mathcal{D} \star \kappa_y$
  → "twisted projectors" $\widetilde{\mathcal{P}}_n = \mathcal{P}_n \star \kappa_y$, satisfying the generalized projector algebra

$$\begin{aligned} \mathcal{P}_n \star \mathcal{P}_m &= \delta_{nm}\mathcal{P}_n \ , & \widetilde{\mathcal{P}}_n \star \widetilde{\mathcal{P}}_m &= \delta_{n,-m}\mathcal{P}_n \\ \mathcal{P}_n \star \widetilde{\mathcal{P}}_m &= \delta_{nm}\widetilde{\mathcal{P}}_n \ , & \widetilde{\mathcal{P}}_n \star \mathcal{P}_m &= \delta_{n,-m}\widetilde{\mathcal{P}}_n \ , \end{aligned}$$

$$\Longrightarrow$$

(C.I., P. Sundell, to appear)

$$\widetilde{\mathcal{P}}_n(E) \ \simeq \ |n/2;0\rangle\langle -n/2;0| \ \in \ \mathcal{D}_0 \otimes \widetilde{\mathcal{D}}_0^*$$

- κ_y flips the sign of energy and thus generates states that lie outside the particle modules , static, with same eigenvalues of static runaway modes but "dual" spacetime behaviour (r → $1/r$) → *soliton*-like solutions.
- In fact, such states provide the local data for HS generalizations of Schwarzschild black holes !

Weyl 0-form: initial data for HS black holes

- $\Phi'(Y)$ expanded in twisted projectors: $\Phi'(Y) \;=\; \sum_n \nu_n \, \mathcal{P}_n(Y) \star \kappa_y \;=\; \sum_n \nu_n \, \widetilde{\mathcal{P}}_n(Y)$

- This expansion enforces the Kerr-Schild property in gauge fields: the latter are reconstructed from curvatures in powers of $(\Phi' \star \kappa_y)^{\star n} = \mathcal{P}^{\star n} = \mathcal{P}$.

- Solutions inherit the symmetries of the projectors

$$\delta\Phi(x|Y) \;=\; -[\epsilon(x|Y), \Phi(x|K)]_{\star,\pi} \;=\; 0 \;\Leftrightarrow\; [\epsilon'(Y), \mathcal{P}_n(K)]_\star \;=\; 0 \;\Rightarrow\; \epsilon'_{\text{r.s.}} \in \mathfrak{c}_{\mathfrak{sp}(4,\mathbb{R})}(K)$$

For a Schwarzschild bh, residual isometry → $\mathfrak{so}(2)_E \oplus \mathfrak{so}(3)_{M_{rs}}$. Projectors are $f(E)$:

$$\mathcal{P}_n(E) \;=\; 4(-1)^{n-\frac{1+\varepsilon}{2}} \, e^{-4E} L^{(1)}_{n-1}(8E) \;=\; 2(-1)^{n-\frac{1+\varepsilon}{2}} \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1}\right)^n e^{-4\eta E}$$

- Indeed, reinstating the x-dependence:

$$\Phi(x|Y) \;=\; L^{-1}(x) \star \Phi' \star \pi(L)(x) \;=\; \sum_n \nu_n \mathcal{N}_n \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1}\right)^n \underbrace{L^{-1}(x) \star e^{-4\eta E} \star L(x) \star \kappa_y}$$

a tower of type-D Weyl tensors of all spins: $\Phi^{(n)}_{\alpha(2s)} \;\sim\; \dfrac{i^{n-1}\mu_n}{r^{s+1}} \, (u^+u^-)^s_{\alpha(2s)}$

(Didenko-Vasiliev, '09, C.I.-P. Sundell, '11)

Weyl 0-form: initial data for AdS massless scalar

- Φ'(Y) expanded on projectors:

$$\Phi'(Y) \;=\; \sum_n \widetilde{\nu}_n \, \widetilde{\mathcal{P}}_n(Y) \star \kappa_y \;=\; \sum_n \widetilde{\nu}_n \, \mathcal{P}_n(E) \;, \qquad (\widetilde{\nu}_n)^* \;=\; \widetilde{\nu}_{-n}$$

$$\widetilde{\mathcal{P}}_n(E) \;:=\; \mathcal{P}_n(E) \star \kappa_y \;=\; 4\pi(-)^{n-\frac{1+\varepsilon}{2}} \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1}\right)^n \delta^2(y - i\eta\sigma_0\bar{y})$$

- Weyl zero-form only contains a scalar (modes of an AdS massless scalar):

$$\Phi(x|Y) \;=\; L^{-1}(x) \star \Phi' \star \pi(L)(x) \;=\; \sum_n \tilde{\nu}_n \mathcal{N}_n \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1}\right)^n \underbrace{L^{-1}(x) \star \delta^2(y - i\eta\sigma_0\bar{y}) \star L(x) \star \kappa_y}$$

$$\Phi(x|Y) \;=\; (1-x^2) \sum_n \mathcal{N}_n \widetilde{\nu}_n \oint_{C(\epsilon)} \frac{d\eta}{2\pi i} \left(\frac{\eta+1}{\eta-1}\right)^n \frac{e^{iy^\alpha M_\alpha{}^{\dot\beta}(x,\eta)\bar{y}_{\dot\beta}}}{1 - 2i\eta x_0 + \eta^2 x^2}$$

$$4\widetilde{\nu}_1 \frac{(1-x^2)}{1 - 2ix_0 + x^2} \;\sim\; \widetilde{\nu}_1 \frac{e^{-it}}{(1+r^2)^{1/2}}$$

(C.I., P. Sundell, to appear)

- Differently from bhs, one does not expect a free scalar to solve the full equations. However, their completion to full solutions is precisely given by the bh sector!

HS black-holes as a "backreaction"

- Indeed, the nonlinear corrections naturally activate the bh sector, since

$$(\Phi' \star \kappa_y)^{\star n} \sim \widetilde{P}^{\star n} = \begin{cases} P \ , & n = 2k \ , \\ \widetilde{P} \ , & n = 2k+1 \end{cases}$$

$$(F')^{\star k} = \left(\sum_n \widetilde{\nu}_n \widetilde{\mathcal{P}}_n \right)^{\star k} = \sum_n \left(\nu_n^{(k)} \mathcal{P}_n + \widetilde{\nu}_n^{(k)} \widetilde{\mathcal{P}}_n \right)$$

$$\begin{aligned} \nu_n^{(k)} &= |\widetilde{\nu}_n|^k \ , \quad k = 2p \\ \widetilde{\nu}_n^{(k)} &= \widetilde{\nu}_n |\widetilde{\nu}_n|^{k-1} \ , \quad k = 2p+1 \end{aligned}$$

- As a consequence, the deformed oscillators S_α (that give rise to nonlinear corrections to the gauge fields) receive contributions from both sectors → bhs arise in this sense as a backreaction, from nonlinear corrections induced by a massless scalar.

$$\begin{aligned} \widehat{\Phi}'(Y,Z) &= \Phi'(Y) = F'(Y) \star \kappa_y = \bar{F}'(Y) \star \bar{\kappa}_{\bar{y}} \ , \\ \widehat{V}'_\alpha(Y,Z) &= \widehat{V}'_\alpha(Y,z) = \widehat{V}'_\alpha(F'(Y),z) = \sum_{k=1}^{\infty} (F'(Y))^{\star k} \star V_\alpha^{(k)}(z) \end{aligned}$$

$$\widehat{V}'_\alpha = 2i \sum_{k=1}^{\infty} \binom{1/2}{k} \left(-\frac{b}{2} \right)^k \int_{-1}^{1} \frac{dt}{(t+1)^2} \frac{(\log(1/t^2))^{k-1}}{(k-1)!} z_\alpha e^{i\frac{t-1}{t+1} w_z} \star (F')^{\star k}$$

Technical caveat

- Working with non-polynomial functions of the oscillators brings about technical subtleties, related to the fact that ⋆-multiplication of them may result in divergencies.
- This may be remedied by sticking to a specific regular presentation, leading to well-defined ⋆-product compositions (being finite as well as compatible with associativity) for all the functions within a given sector.
- So mixing different sectors of Weyl zero-form moduli also raises the interesting question of whether or not there exists a common regular presentation, one that leads to regular ⋆-products for solutions of both sectors.
- It turns out that the integral presentation indeed works as such a common regular presentation for both the bh and the massless particle sector.
- Subtlety with invariants, $\mathrm{Str}(\mathcal{P}_n \star \kappa_y) = \mathcal{P}_n \star \kappa_y|_{Y=0} = \infty$. However, it is possible to regularize it by using the integral presentation to the unique possible finite value → 0.

Cylindrically-symmetric type-D solutions

- Possible to construct projectors $\mathcal{P}_n$ (K) with any of the generators K ={ E, J, iB, iP }

  ➔ Solutions with $\mathfrak{so}(2,1)_{\mathfrak{h}(K)} \oplus \mathfrak{so}(2)_K$ symmetry .

- In particular, for K = J , $$\mathcal{P}_1(J) \;:=\; 4e^{-\frac{1}{2}Y^{\underline{\alpha}}K'_{\underline{\alpha\beta}}Y^{\underline{\beta}}} \;=\; 4e^{-4J}$$

  Again a ground state of a 2D Fock-space (a non-compact ultra-short irrep, singleton-like but with roles of E and J exchanged, |E| < |J| instead of |E| > |J|).

- Same steps yield $$\Phi(x|Y) \;=\; \sum_n \nu_n \mathcal{N}_n \oint_{C(\varepsilon)} \frac{d\eta}{2\pi i}\left(\frac{\eta+1}{\eta-1}\right)^n \underbrace{L^{-1}(x) \star e^{-4\eta J} \star L(x) \star \kappa_y}$$

$$\Phi^{(n)}_{\alpha(2s)} \;\sim\; \frac{i^{n+s+1}\mu_n}{(1+r^2\sin^2\theta)^{(s+1)/2}}\,(\tilde{u}^+\tilde{u}^-)^s_{\alpha(2s)}$$

II. Z-space monodromies and S-moduli

- The structure of the vacuum $\Phi = 0$ may be richer than it seems at first sight. Indeed, the deformed oscillator equation

$$[\widehat{S}^{(0)}_{\alpha}, \widehat{S}^{(0)\alpha}]_{\star} \;=\; -4i$$

has the structure of $X^2 = 1$. If X is valued in a purely commutative algebra → $X = \pm 1$. However, if it is vaued in a non-commutative algebra, there are more interesting possibilities:

$$X^2 = 1 \;\rightarrow\; X = 1 - 2P \;, \quad P^2 = P$$

- *Projectors in Z-space connection* → flat but non-trivial! More complicated vacua?

$$\widehat{\Phi}' \;=\; 0 \,, \quad \widehat{S}'_{\alpha} \;=\; z_{\alpha} \left(1 - 2 \sum_{n=0}^{\infty} \theta_n P_n(Y,Z) \right) \,, \qquad P_n \star P_m \;=\; \delta_{nm} P_n \,, \qquad \theta_n \;=\; \{0,1\}$$

(C.I.-Sezgin-Sundell '07)

➢ Can also dress up non-vacuum solutions (like windings in String Theory…). Both bhs and the scalar-instanton can be decorated with these discrete S-moduli.

➢ Different choices for the Fock-space where the projectors act lead to different global symmetries.

Φ-moduli and S-moduli of so(3,1)-invariant solutions

- Local data for full $\mathfrak{so}(3,1)$-invariant solution: *(C.I., E.Sezgin, P. Sundell, '07)*

$$\begin{aligned}
\hat{\Phi}' &= \nu\ , \quad \nu \in \mathbf{R}\ ; \qquad \hat{A}'_\alpha = \hat{A}'^{(reg)}_\alpha + \hat{A}'^{(proj)}_\alpha \\
\hat{A}'^{(reg)}_\alpha &= \frac{i\nu}{8} z_\alpha \int_{-1}^{1} dt\, e^{\frac{i}{2}(1+t)u} \left[{}_1F_1\left(1/2; 2; \frac{\nu}{2}\log\frac{1}{t^2}\right) + t\ {}_1F_1\left(1/2; 2; -\frac{\nu}{2}\log\frac{1}{t^2}\right)\right] \\
\hat{A}'^{(proj)}_\alpha &= -iz_\alpha \sum_{k=0}^{\infty} (-1)^k \theta_k L_k[\nu] P_k(u)\ , \quad P_k(u) = \frac{1}{k!}\left(\frac{-iu}{2}\right)^k e^{iu/2}\ , \qquad u := y^\alpha z_\alpha
\end{aligned}$$

$$L_k[\nu] = (-1)^k - \frac{1+(-1)^k}{2}\left(1 - \sqrt{1 - \frac{c_1\nu}{1+k}}\right) - \frac{1-(-1)^k}{2}\left(1 - \sqrt{1 + \frac{c_1\nu}{2+k}}\right)$$

Physical fields:

1) $\theta_k = 0\ , \quad \forall k$

- **0-form:** only scalar field
- **gauge fields:** only rescaled AdS metric

$$\begin{aligned}
\phi(x) &= \nu(1-x^2) \\
ds^2 &= \frac{4\Omega^2 (d(g_1 x))^2}{(1-g_1^2 x^2)^2}
\end{aligned}$$

$$\lim_{\lambda^2 x^2 \to 1} \Omega, g_1 = 1$$

2) $\nu = 0$, $(\theta_k - \theta_{k+1})^2 = 1$

- **gauge fields:** degenerate metric

$$ds^2 = \frac{4(dx^\mu x_\mu)^2}{\lambda^2 x^2 (1-\lambda^2 x^2)}$$

Conclusions & Outlook

- A relatively vast realm of exact solutions of 4D Vasiliev's equations has been found, by various methods, and they are promtping us to better understand the peculiarities of HS theories, both technical and conceptual.

- How to extract the most relevant physical data (e.g., asymptotic charges, thermodynamics, phase transitions,…) ?

- How to physically characterize the solutions? Important to better understand and evaluate various HS invariants. Behaviour of certain observables on the solutions may probe properties of the theory (e.g., existence of multi-body solutions?)

- Important related issues concerning functional classes of twistor-space elements, and related questions of the admissibility of gauge parameters, regularity under star-product, regularity of observables,…