Remarks on conformal invariance of gauge fields

Maxim Grigoriev

Lebedev Physical Institute, Moscow

Based on:

G. Barnich, X. Bekaert, M.G., arXiv:1506.00595

some earlier related works with G. Barnich, X. Bekaert, K. Alkalaev, I. Tipunin

HSTS-2, June 4, Moscow

General question: classification of symmetries of a given system

- PDE symmeries *Vinogradov; Anderson; Dickey*

- Conformal differential operators *Penrose; Eastwood,Rice; Dobrev*

- HS fields *Eastwood; Vasiliev; Anco, Pohjanpelto; Shaynkman, Tipunin, Vasiliev; Bekaert, M.G.*

Typical particular question: whether a system invariant under Lie algebra $\mathfrak{h}$ admits a bigger algebra $\mathfrak{g} \supset \mathfrak{h}$

Main example $iso(d-1,1)$ (Poincaré) or $so(d-1,1)$ (AdS_d) extends to $so(d,2)$ (conformal).

Two approaches (closely related though not equivalent in general):

1) Group theoretical: whether an (irreducible) $\mathfrak{h}$-module of suitable class of solutions admits a lift to $\mathfrak{g}$-module.

$\mathfrak{h} = iso(d-1,1)$, $\mathfrak{g} = so(d,2)$ *Gross 1964, Bracken, Jessup 1982, Siegel 1988,*

$\mathfrak{h} = so(d-1,2)$, $\mathfrak{g} = so(d,2)$ *Metsaev 1995*

2) EOM symmetries: whether an $\mathfrak{h}$-symmetry of EOMs extends to $\mathfrak{g} \supset \mathfrak{h}$

Plan:

- Generalities
  - Linear symmetries for non-gauge EOM's
  - Linear symmetries of gauge EOM's
  - BRST complex
  - Relation to unfolded approach
  - Potentials vs. curvatures
- Obstructions
  - Global reducibility parameters
  - Branching rules
- Examples
  - Fronsdal fields
  - Maximal depth partially-massless (PM) fields
  - PM vs. FT fields
- Variational symmetries
- Conclusions

Linear EOM's, x^μ space-time coordinates (independnet variables), $\phi^i(x)$ fields (dependent variables)

$$T^a_i(x, \frac{\partial}{\partial x})\phi^i = 0$$

Linear symmetry: $A^i_j(x, \frac{\partial}{\partial x})$ that maps solutions to solutions. $A^i_j(x, \frac{\partial}{\partial x})$ is trivial if $A^i_j\phi^j_0 = 0$ for any solution ϕ_0.

Assuming T^a_i are regular (i.e. any trivial A has the form $\overset{(1)}{B}{}^i_a T^a_j$) the definition:

$$T^a_i A^i_j = \overset{(1)}{A}{}^a_b T^b_j \qquad A^i_j \sim A^i_j + \overset{(1)}{B}{}^i_a T^a_j$$

Standard definition, e.g. *Eastwood, 2002*

Linear gauge systems

In addition to $T_i^a(x, \frac{\partial}{\partial x})$ there are gauge generators $R^i_\alpha(x, \frac{\partial}{\partial x})$ such that

$$T_i^a(x, \frac{\partial}{\partial x}) R^i_\alpha(x, \frac{\partial}{\partial x}) = 0 \qquad \text{i.e. } R^i_\alpha \chi^\alpha \text{ is always a solution (pure gauge)}$$

Solutions differing by a pure gauge are considered equivalent.

Linear global symmetries of a gauge system: a global symmetry should respect equivalence classes:

$$A^i_j R^j_\alpha = R^i_\beta \overset{(-1)}{A}{}^\beta_\alpha \,, \qquad T^a_i A^i_j = \overset{(1)}{A}{}^a_b T^b_j$$

Symmetries proportional to gauge ones are also trivial i.e.

$$A^i_j \sim A^i_j + \overset{(1)}{B}{}^i_a T^a_j + R^i_\alpha \overset{(0)}{B}{}^\alpha_j$$

This gives a definition of global symmetry of a gauge system if T^i_a and R^i_α are independent (irreducible system).

Generalization is easily achieved through the following reformulation:

Linear spaces

$$\mathcal{H}_{-1} = \{\chi^\alpha(x)\}\,, \qquad \mathcal{H}_0 = \{\phi^i(x)\}\,, \qquad \mathcal{H}_1 = \{\phi^a_*(x)\}$$

Consider degree 1 operator Ω:

$$\Omega: \quad \overset{\text{(i)}}{\Omega} : \mathcal{H}_i \to \mathcal{H}_{i+1} \qquad \overset{\text{(-1)}}{\Omega} = R^i_\alpha(x, \frac{\partial}{\partial x})\,, \quad \overset{\text{(0)}}{\Omega} = T^a_i(x, \frac{\partial}{\partial x})\,, \quad \overset{\text{(1)}}{\Omega} = 0$$

The compatibility $TR = 0$ is contained in $\Omega^2 = 0$ and regularity in $H^1(\Omega) = 0$.

Consider degree 0 opeartor $\mathbf{A}$ and degree -1 opearator $\mathbf{B}$

$$\mathbf{A}: \qquad \overset{\text{(i)}}{\mathbf{A}} : \mathcal{H}_i \to \mathcal{H}_i \qquad \overset{\text{(i)}}{\mathbf{A}} = \overset{\text{(i)}}{A}\,, \quad \overset{\text{(0)}}{A}{}^i_j = A^i_j$$

$$\mathbf{B}: \qquad \overset{\text{(i)}}{\mathbf{B}} : \mathcal{H}_i \to \mathcal{H}_{i-1} \qquad \overset{\text{(i)}}{\mathbf{B}} = \overset{\text{(i)}}{B}$$

The definition of symmetry becomes: *Barnich, M.G., Semikhatov, Tipunin 2004*

$$[\Omega, \mathbf{A}] = 0\,, \qquad \mathbf{A} \sim \mathbf{A} + [\Omega, \mathbf{B}]$$

i.e. "BRST first-quantized observables" $H^0([\Omega, \cdot])$

General linear gauge system:

$$(\Omega, \mathcal{H}), \qquad \mathcal{H} = \oplus_i \mathcal{H}_i, \qquad deg(\Omega) = 1, \quad \Omega^2 = 0 \quad H^i(\Omega) = 0 \ \forall \ i > 0$$

Technical assumption: $H^i(\Omega)$ is finite dimensional for $i < 0$. (First quantized) BRST.

Global symmetries:

$$[\Omega, \mathbf{A}] = 0, \qquad \mathbf{A} \sim \mathbf{A} + [\Omega, \mathbf{B}]$$

Linear gauge EOMs $\Leftrightarrow$ (Formal) quantum constrained systems

We have (almost obvious) statements:

- $\mathcal{A} = H^0([\Omega, \cdot])$ – associative algebra of inequivalent global symmetries
- $H^i([\Omega, \cdot])$ – representation of $\mathcal{A}$ or any its (Lie) subalgebra
- $H^i(\Omega, \mathcal{H})$ – representation of $\mathcal{A}$ or any its (Lie) subalgebra

In particular, if the system is by construction $\mathfrak{h}$-invariant all $H^i(\Omega, \mathcal{H})$ are $\mathfrak{h}$-modules.

If $\mathfrak{h}$-symmetry lifts to $\mathfrak{g} \supset \mathfrak{h}$ then all $H^i(\Omega)$ can be lifted to $\mathfrak{g}$-modules.

This gives a powerful (as we are going to see) necessary condition for a system to admit $\mathfrak{g}$-symmetry.

For instance to show that system does not admit $\mathfrak{g}$ symmetry it is enough to show that at least one $H^i(\Omega)$ is not an $\mathfrak{g}$-module.

For non-gauge system and $H^0(\Omega)$ in a suitable functional space this reproduces the standard approach *. . . ,Bracken, Jessup 1982,Siegel 1988, Metsaev 1995, Shaynkman, Tipunin, Vasiliev 2003,. . .*

For gauge system $H^i(\Omega)$ with $i < 0$ are typically finite-dimensional so that classical representation theory applies; hence the first thing to check.

For instance $H^{-1}(\Omega)$ is the space of global reducibility parameters: $R^i_\alpha \chi^\alpha_0 = 0$, e.g. Killing tensors. For $i < -1$ their higher analogs associated to higher reducibility relations.

(Non)Branching rules:

In application we can often concentrate on an irreducible system. In this case at least one (but usually all) $H^i(\Omega)$ are irreducible $\mathfrak{h}$-modules. If the $\mathfrak{h}$-symmetry lifts to $\mathfrak{g} \supset \mathfrak{h}$ symmetry then

$$(H^i(\Omega))_{\mathfrak{g}} \downarrow (H^i(\Omega))_{\mathfrak{h}}$$

i.e. as an $\mathfrak{g}$-module $H^i(\Omega)$ does not branch. For instance:

Lemma 1. *A nontrivial irreducible $so(d+2)$-module $\mathcal{D}_{o(d+2)}(s_1, \ldots, s_r)$ remains irreducible after its restriction to $so(d+1)$ if and only if*
d is even $r = \frac{d+2}{2}$ and $s_1 = \ldots = s_{r-1} = |s_r|$,
i.e., if it is described by a rectangular Young Tableaux (YT) of height $\frac{d+2}{2}$.

As an $o(d+1)$ module it is associated to rectangular YT of height $\frac{d}{2}$.

Example: rank 3 selfdual antisymmetric tensors in 6d are 1:1 with rank 2 antysymmetric tensors in 5d.

Relation with unfolded approach

Unfolded formulation *Vasiliev 1988,Lopatin, Vasiliev 1988, Vasiliev 2005,...*

In the minimal unfolded formulation $H^i(\Omega)$ are realized manifestly. More precisley p-form fields take values in $H^{-p}(\Omega)$. The relation is best understood using the parent construction *Barnich, M.G. Semikhatov, Tipunin 2004, Barnich M.G. 2006,2010*.

Parent formulation

$$\Omega^P = dx^\mu \left(\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial y^\mu}\right) + \bar{\Omega}\,, \qquad \bar{\Omega} = \Omega|_{x^\mu \to x^\mu + y^\mu, \frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial y^\mu}}\,. \tag{1}$$

new variables y^μ, new constraints $(\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial y^\mu})\Phi = 0$, new ghosts dx^μ

Consider $H^{-p}(\bar{\Omega}_x)$ at fixed x^μ. This is $H^{-p}(\Omega)$ in formal power series around x. If $H^{-p}(\bar{\Omega}_x)$ is isomorphic for all x^μ (this happens for $\mathfrak{h}$-invariant systems on H-cosets) one can eliminate all fields but those associated to $H^i(\bar{\Omega})$. EOM's and gauge transformations

$$\nabla\phi_0 = 0\,, \quad \nabla\phi_1 + \sigma_2\phi_0 = 0\,, \quad \nabla\phi_2 + \sigma_2\phi_1 + \sigma_3\phi_0 = 0\,, \quad \ldots$$
$$\delta\phi_0 = 0\,, \quad \delta\phi_1 = \nabla\chi_1\,, \quad \delta\phi_2 = \nabla\chi_2 + \sigma_2\chi_1\,, \quad \ldots$$

In general, $H^{-p}(\bar{\Omega}_x)$ may be different at different x. E.g. ambient space formulation of AdS field. $H^0(\bar{\Omega}_X)$ on the cone $X^2 = 0$ and the hyperbolid $X^2 = -1$ are not isomorphic (though related). This underlies ambient approach to boundary values of *Bekaert, M.G. 2012, 2013*.

Note that $H^{-p}(\bar{\Omega}_x)$ is unchanged for $p > 0$.

Potentials vs. curvatures

- In the unfolded formulation one can consistently put $\phi_p = 0$ for $p > 0$, giving $\nabla\phi_0 = 0$.

This gives a non-gauge system such that $H^0(\Omega') = H^0(\Omega')$ (i.e. the spaces of gauge inequivalent solutions coincide). Formulation in terms of curvatures.

Usually the curvature formulation is constructed from scratch without using above techniques. For instance, for e.g. Fronsdal fields, Fradkin Tseytlin fields curvature formulations are known for quite a while *Weinberg 1965, Fradkin, Tseytlin 1985,...*

The fundamental field is (generalized) Weyl tensor.

The linear gauge system $(\Omega, \mathcal{H})$ and its curvature formulation are not equivalent as local field theories and e.g. may have different global symmetries. E.g. Fronsdal fields in 4d are not conformal for $s > 1$ but its curvature formulation is. The difference can e.g. be traced to $H^{-p}(\Omega)$ for $p > 0$.

We are forced to work with potentials due to Interactions, Lagrangian, Quantization

(Warm-up) example: Fronsdal fields in flat space

Generating functions:

$$\Phi(x,p) = \phi_{\mu_1\ldots\mu_s}\, a^{\mu_1}\ldots a^{\mu_s} \qquad \chi = \chi_{\mu_1\ldots\mu_{s-1}}\, a^{\mu_1}\ldots a^{\mu_{s-1}}$$

satisfying

$$\frac{\partial}{\partial x}\cdot\frac{\partial}{\partial x}\Phi = \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial a}\Phi = \frac{\partial}{\partial a}\cdot\frac{\partial}{\partial a}\Phi = 0\,, \quad \text{same for } \chi$$

Gauge transformation

$$\delta_\chi \Phi = a\cdot\frac{\partial}{\partial x}\chi$$

$H^{-1}(\Omega)$ – tracelss Killing tensors:

$$a\cdot\frac{\partial}{\partial x}\chi_0 = 0\,, \qquad \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial x}\chi_0 = \frac{\partial}{\partial x}\cdot\frac{\partial}{\partial a}\chi_0 = \frac{\partial}{\partial a}\cdot\frac{\partial}{\partial a}\chi_0 = 0$$

The Poincaré translations act like $P_\mu = \frac{\partial}{\partial x^\mu}$. If it lifts to conformal the later should arise from generalized Verma induced from $V_0 \subset H^{-1}(\Omega)$ annihilated by P_μ. V_0 is a Lorentz module $\mathcal{D}(s-1,0,\ldots,0)$.

If we are after variational symmetries then $\Delta = \frac{d}{2} - 2$ for V_0 which for $d \geqslant 0$ implies $s = 1$. Even no need to use $H^0(\Omega)$!

At the EOM level one also gets $s = 1$ but to get $d = 4$ one employs known results on $H^0(\Omega)$ e.g. *Shaynkman, Tipunin, Vasiliev 2003*.

The procedure works similiarly for mixed-symmetry fields. $H^{-p}(\Omega)$ known from *Alkalaev, M.G. Tipunin, 2008* (also *Skvortsov 2008*). Again only totally antisymmeric field of rank $d/2 - 1$ is conformal in terms of potential.

But in terms of curvatures fields associated with rectangular YT of height $\frac{d}{2} - 1$ and arbitrary length are conformal for even d.

Maximal-depth PM fields in 4d

Partially massless fields *Deser, Nepomechie 1983; Deser, Waldron 2001; Skvortsov Vasiliev 2006;. . .*

Ambient space X^B, $(B = 0, \cdots, 4)$, flat metric $\eta_{AB} = \text{diag}(-,+,+,+,-)$, anti-de Sitter spacetime AdS_4 is the hyperbolid $X \cdot X + 1 = 0$.

In terms of generating function:

Alkalaev M.G. 2009,2011, M.G. Waldron 2011, Joung, Lopez,Taronna 2012

$$(X \cdot \frac{\partial}{\partial X} + 1)\Phi = 0\,, \qquad X \cdot \frac{\partial}{\partial A}\Phi = 0\,, \qquad (A \cdot \frac{\partial}{\partial A} - s)\Phi = 0\,,$$
$$\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X}\Phi = \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial A}\Phi = \frac{\partial}{\partial A} \cdot \frac{\partial}{\partial A}\Phi = 0\,,$$

gauge transformations with $\chi = \chi(X)$

$$\delta_\chi \Phi = (A \cdot \frac{\partial}{\partial X})^s \chi\,, \quad (X \cdot \frac{\partial}{\partial X} - s + 1)\chi = 0\,, \quad \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X}\chi = 0\,.$$

As a $so(3,2)$-module the global reducibilitites $H^{-1}(\Omega) = \mathcal{D}(s-1, 0)$. As we have seen such module does not lift to $o(4,2)$. Not conformal in terms of potentials.

Curvatures

Still one needs to check if similarly to Fronsdal fields in 4d maximal-depth PM fields are not conformal in terms of curvatures. As Lorentz $so(3,1)$ tensor the curvature is $F_{\mu|\nu_1\ldots\nu_s}$ and associated to $\mathcal{D}(s,1)$. If it were conformal its Weyl transformation to flat space should coincide with naive flat limit if it's regular.

In flat space the powerful method is available *Shaynkman, Tipunin, Vasiliev 2003*. This gives possible relevant conformal equations: the one with Lorentz tensor structure $\mathcal{D}(s,0)$ and conformal dimension $\Delta = 2$ for F is. There are no more 1st order equations for F of the same Δ.

maximal depth FT fields

The conformal equation $\mathcal{D}(s,0)$ corresponds to curvature formulation of a very similar system: conformal symmetric tensors (CST) or maximal-depth Fradkin-Tseytlin (FT) field *cf. talk by A.Tseytlin*

- $s=1$ Maxwell, $s=2$ *Deser, Nepomechie 1983*
- $s>2$ *Erdmenger,Osborn 1997*
- Belongs to the class of conformal gauge fields (generalizing FT ones) *Vasiliev 2009*
- Boundary values of maximal-depth PM fields in AdS_5 *Bekaert, M.G. 2013*

Lagrangian *Erdmenger,Osborn 1997*

$$L=\partial^\nu\varphi^{\mu_1\dots\mu_s}\partial_\nu\varphi_{\mu_1\dots\mu_s}-\frac{2s}{s+1}\partial_\nu\varphi^{\nu\mu_2\dots\mu_s}\partial^\lambda\varphi_{\lambda\mu_2\dots\mu_s}$$

gauge transformations: $\delta\varphi_{\mu_1\dots\mu_s}=\partial_{\mu_1}\dots\partial_{\mu_s}\chi-$ traces.

$s=2$ PM field in terms of curvatures

The curvature *Deser, Waldron 2006*

$$F_{\mu\nu|\rho} = \nabla_\mu \varphi_{\nu\rho} - \nabla_\nu \varphi_{\mu\rho}\,.$$

EOM's

$$\nabla^\rho F_{\rho(\mu|\nu)} - g_{\mu\nu}\nabla^\rho F'_\rho + \nabla_{(\mu}F'_{\nu)} = 0\,,$$

where $F'_\mu = F_{\mu\rho|\nu}g^{\rho\nu}$ and $X_{(a}Y_{b)} = \frac{1}{2}(X_aY_b + X_bY_a)$.

Lagrangian *Deser, Waldron 2006*

$$L^{PM} = F_{\mu\nu|\rho}F^{\mu\nu|\rho} + F'^{\,\nu}F'_\nu\,.$$

If $F_{\mu\nu|\rho}$ is fundamental, the complete system

$$F_{\mu\nu|\rho} = -F_{\nu\mu|\rho}\,, \qquad F_{[\mu\nu|\rho]} = 0\,, \qquad F_{\mu\nu|\rho}g^{\nu\rho} = 0\,,$$
$$\nabla^\mu F_{\mu\nu|\rho} = 0\,, \qquad \nabla_{[\sigma}F_{\mu\nu]|\rho} = 0\,.$$

$s = 2$ maximal depth FT field in 4d in terms of curvatures

The traceless component of the curvature is

$$\widetilde{F}_{\mu\nu|\rho} = \nabla_\mu \varphi_{\nu\rho} - \nabla_\nu \varphi_{\mu\rho} - \frac{1}{3} g_{\mu\rho} \nabla^\alpha \varphi_{\alpha\nu} + \frac{1}{3} g_{\nu\rho} \nabla^\alpha \varphi_{\alpha\mu} \,.$$

EOM's and Lagrangian

$$\nabla^\mu \widetilde{F}_{\mu(\nu|\rho)} = 0 \,, \quad L^{FT} = \frac{1}{2} \widetilde{F}_{\mu\nu|\rho} \widetilde{F}^{\mu\nu|\rho} \,.$$

If one treats $\widetilde{F}_{\mu\nu|\rho}$ as the fundamental fields, the complete set of equations is

$$\widetilde{F}_{\mu\nu|\rho} = -\widetilde{F}_{\nu\mu|\rho} \,, \qquad \widetilde{F}_{[\mu\nu|\rho]} = 0 \,, \qquad \widetilde{F}_{\mu\nu|\rho} g^{\nu\rho} = 0 \,,$$
$$\nabla^\mu \widetilde{F}_{\mu(\nu|\rho)} = 0 \,, \qquad \nabla_{[\sigma} \widetilde{F}_{\mu\nu]|\rho} = g_{\rho[\sigma} A_{\mu\nu]} \,,$$

- Algebraic consraints coincide with PM ones
- Differential equations are a subset of PM ones
- PM solutions form an $o(3,2)$-submodule of FT ones (but not $o(4,2)$)

General AdS gauge fields.

For a generic AdS_d field the only nonvanishing $H^i(\Omega)$ is $H^{-p}(\Omega)$, where p is a position the raw in the associated YT to which gauge transformations symmetry is associated *Alkalaev, M.G. 2009,2011*

Can be also inferred from earlier works *Alkalaev, Shaynkman, Vasiliev 2003, Skvortsov 2009*.

For instance candidates are Fronsdal fields in 4d. In this case $H^{-1}(\Omega) = \mathcal{D}(s-1, s-1)$ (the familiar 2-raw YT). However, the self-duality in $4+2$ dimensions does not have real solutions. Though there is a chance with complex (i.e. doubled set of) fields *cf. Vasiliev 2001*.

More generally, one can rule out all PM fields just by considering $H^{-p}(\Omega)$.

Variational symmetries

If linear gauge system $\Omega, \mathcal{H}$ is Lagrangian $\mathcal{H}$ is equipped with graded symmetric inner product

$$\langle \phi, \chi \rangle = \int d^d x g_{ij} \phi^i(x) \chi^j(x)\,, \quad \langle \phi, \chi \rangle = (-1)^{|\phi||\chi|} \langle \chi, \phi \rangle\,, \quad \mathrm{gh}(\phi) + \mathrm{gh}(\chi) = 1$$

such that $\Omega^\dagger = \Omega$. The action

$$S = \frac{1}{2} \langle \phi_0, \Omega \phi_0 \rangle \quad \mathrm{gh}(\phi_0) = 0\,, \qquad S_{BV} = \frac{1}{2} \langle \Psi, \Omega \Psi \rangle$$

Variational global symmetries

$$A^\dagger = -A\,, \qquad [\Omega, \mathbf{A}] = 0\,, \qquad \mathbf{A} \sim \mathbf{A} + [\Omega, \mathbf{B}]$$

Note that variational symmetries form Lie rather than associative algebra.

In fact $\langle , \rangle$ determines a Batalin-Vilkovisky antibracket, $S_{BV} = \frac{1}{2} \Psi \Omega \Psi$, Global symmetry generators $\frac{1}{2} \langle \Psi, IA\Psi \rangle$, $Ie_a = (-)^{|a|} e_a$.

Simplest example: Varational symmetries of KG equation

All symmetries are $1:1$ with conformal Killing tensors *Eastwood 2002*

Variational symmetries are $1:1$ with odd-rank conformal Killing tensors. An associated Noether currents are then even-rank.

Conclusions

- Simple though powerful necessary condition for global symmetries of gauge systems in terms of (higher-order) global reducibility parameters.

- Global reducibility parameters are tightly related to so-called surface charges (Local BRST cohomology degree $d-2$ (and hence represented by $d-2$-forms).

- Interesting to classify all variational symmetries of real $d/2-1$ antisymmetric fieds. Proposal for all symmetries *Bekaert, M.G. 2009*. Extra selection criteria for HS algebras.