Symmetries of $4d$ higher-spin current interactions

O.A.Gelfond

O.G., M.Vasiliev (to appear)

ISR and Lebedev Institute of RAS, Moscow

Higher Spin Theory and Holography-2

Lebedev Institute, Moscow, June 4, 2015

Plan

Introduction

Infinite towers of free massless fields in the $4d$ HS gauge theory exhibit $\mathfrak{sp}(8) \supset \mathfrak{su}(2,2)$. C. Fronsdal (1986)

Manifestly $\mathfrak{sp}(8)$ symmetric geometric realization of field equations of massless fields of all spins was actively studied I. Bandos and J. Lukierski, (1999) ; I. Bandos, J. Lukierski and D. Sorokin, (2000); M.A. Vasiliev, (2001), (2002), (2008); M. Plyushchay, D. Sorokin and M. Tsulaia, (2003); V. E. Didenko and M. A. Vasiliev, (2004); S. Fedoruk and J. Lukierski, (2013); I. Florakis, D. Sorokin and M. Tsulaia, (2014)...

Attempts to extend the formalism to HS interactions I. Bandos, X. Bekaert, J. A. de Azcarraga, D. Sorokin and M. Tsulaia, (2005)

Full nonlinear system of HS equations is not manifestly $\mathfrak{sp}(8)$ symmetric.

HS interactions are shown to necessarily break $\mathfrak{sp}(8)$ down $\mathfrak{su}(2,2)$.

Unfolded description of current HS interactions can be understood as a deformation of the two independent $4d$ linear systems for rank-one fields and rank-two currents . Each of these systems is $\mathfrak{sp}(8)$ symmetric, but $\mathfrak{sp}(8)$ is not preserved by the deformation.

Unfolded dynamics

Unfolded dynamics controls symmetries in a system M.Vasiliev (1988)

Unfolded formulation of a linear or nonlinear system of partial differential equations and/or constraints in a d-dimensional manifold $M^d(x^{\underline{n}})$ $(\underline{n} = 0, 1, \ldots d-1)$

$$\mathrm{d}W^\Phi(x) = G^\Phi(W(x)) \,, \qquad \mathrm{d} = dx^{\underline{n}} \frac{\partial}{\partial x^{\underline{n}}}$$

$W^\Phi(x)$: set of degree p_Φ-differential forms.

$$G^\Phi(W) = \sum f^\Phi{}_{\Omega_1 \ldots \Omega_n} W^{\Omega_1} \wedge \ldots \wedge W^{\Omega_n}$$

$f^\Phi{}_{\Omega_1 \ldots \Omega_n}$ - structure coefficients.

Generalized Jacobi identity $G^\Omega(W) \wedge \frac{\partial G^\Phi(W)}{\partial W^\Omega} = 0$

Gauge transformation $\delta W^\Phi(x) = \mathrm{d}\varepsilon^\Phi(x) + \varepsilon^\Omega(x) \frac{\partial G^\Phi(W(x))}{\partial W^\Omega(x)}$

Parameter $\varepsilon^\Phi(x)$ is a $(p_\Phi - 1)$-form

Vacuum

w^α : only one-forms $\Rightarrow$ Unfolded equations $=$ Flatness condition

$$\mathrm{d}w^\alpha + \frac{1}{2} f^\alpha_{\beta\gamma} w^\beta \wedge w^\gamma = 0$$

General Jacobi identity $=$ Jacobi identity $\Rightarrow$ $f^\alpha_{\beta\gamma}$ defines Lie algebra $\mathfrak{g}$.

Gauge transformation

$$\delta w^\alpha(x) = D\varepsilon^\alpha(x) := \mathrm{d}\varepsilon^\alpha(x) + f^\alpha_{\beta\gamma} w^\beta(x) \varepsilon^\gamma(x)\,.$$

A flat connection $w(x)$ is invariant under the gauge transformation with the covariantly constant parameters

$$D\varepsilon^\alpha(x) = 0$$

$\varepsilon^\alpha(x)$ can be reconstructed in terms of $\varepsilon^\alpha(x_0)$ at any given point x_0

$\varepsilon^\alpha(x_0)$ are the parameters of the global symmetry $\mathfrak{g}$.

w^α- vacuum flat connection of the Lie algebra $\mathfrak{g}$.

Background geometry is coordinate independent.

Linearized unfolded equations

In the perturbative analysis, w^{α} is assumed to be of the zeroth order.

Differential forms $W^{\Phi} = w^{\Phi} + \omega^{\Phi}$ are small perturbations around w.

If $\omega^{i}(x)$ have a given degree $p_i \Rightarrow G^{i} = -w^{\alpha}(T_{\alpha})^{i}{}_{j} \wedge \omega^{j}$.

General Jacobi identities $\Rightarrow$ $(T_{\alpha})^{i}{}_{j}$ form a representation T of $\mathfrak{g}$

in the space V where $\omega^{i}(x)$ are valued.

Linearized unfolded equation= Covariant constancy condition

$$D\omega^{i} := \mathrm{d}\omega^{i} + w^{\alpha}(T_{\alpha})^{i}{}_{j} \wedge \omega^{j} = 0$$

D is a covariant derivative in the $\mathfrak{g}$-module V.

Once the vacuum connection is fixed, this equation is invariant

under the global symmetry $\mathfrak{g}$ with the covariantly constant parameters

$$\delta\omega^{i}(x) = -\varepsilon^{\alpha}(x)(T_{\alpha})^{i}{}_{j}\omega^{j}(x)\,.$$

$\mathfrak{g}$-invariant linear system of partial differential equations

is reformulated in terms of $\mathfrak{g}$-modules

AdS_4

$\mathfrak{sp}(4,\mathbb{R})$ connection $w^{AB} = w^{BA}$ satisfies $\mathfrak{sp}(4,\mathbb{R})$ zero curvature conditions

$$R^{AB} = 0\,, \qquad R^{AB} = dw^{AB} + w^{AC} \wedge w_C{}^B\,,$$

$$A_B = A^A \mathsf{C}_{AB}\,, \qquad A^A = \mathsf{C}^{AB} A_B\,, \qquad \mathsf{C}_{AC}\mathsf{C}^{BC} = \delta_A^B\,,$$

$\mathsf{C}_{AB} = -\mathsf{C}_{BA}$: invariant form.

Two-component spinor notations

AdS_4 dynamics is described by the Lorentz connection $\omega^{\alpha\beta}$, $\overline{\omega}^{\alpha'\beta'}$ and vierbein $e^{\alpha\alpha'}$.

AdS_4 : space with matrix coordinates $x^{\alpha\alpha'} = x^{\underline{n}}\sigma_{\underline{n}}^{\alpha\beta'}$ and auxiliary commuting spinor variables y^α and $\bar{y}^{\alpha'}$

$\sigma_{\underline{n}}^{\alpha\beta'}$ Hermitian 2×2 matrices, $\qquad \underline{n} = 0,1,2,3,\ \ \alpha,\beta = 1,2$ and $\alpha',\beta' = 1,2$

Unfolded equations of massless fields in AdS_4 M.Vasiliev (1989)

$$D^{ad}\omega(y,\bar{y}|x) = e_\alpha{}^{\alpha'} \wedge e^{\alpha\beta'}\bar{\partial}_{\beta'}\bar{\partial}_{\alpha'}\overline{C}(0,\overline{y}\mid x) + e^\alpha{}_{\alpha'} \wedge e^{\beta\alpha'}\partial_\beta\partial_\alpha C(y,0\mid x)$$

$$D^{tw}C(y,\bar{y}|x) = 0$$

$\omega(y,\bar{y}|x)$ 1$-$forms, $\qquad C(y,\bar{y}|x)$ 0$-$forms , $\qquad \partial_\beta = \frac{\partial}{\partial y^\beta}\,, \qquad \bar{\partial}_{\alpha'} = \frac{\partial}{\partial \bar{y}^{\alpha'}}$

Adjoint derivative

$$D^{ad}\omega(y,\bar{y}|x) = D^L\omega(y,\bar{y}|x) - \lambda e^{\alpha\beta'}\Big(y_\alpha\bar{\partial}_{\beta'} + \partial_\alpha\bar{y}_{\beta'}\Big)\omega(y,\bar{y}|x)\,, \qquad (D^{ad})^2 = 0$$

Twisted adjoint derivative

$$D^{tw}C(y,\bar{y}|x) = D^LC(y,\bar{y}|x) + \lambda e^{\alpha\beta'}\Big(y_\alpha\bar{y}_{\beta'} + \bar{\partial}_{\beta'}\partial_\alpha\Big)C(y,\bar{y}|x)\,, \qquad (D^{tw})^2 = 0$$

Lorentz covariant derivative

$$D^LA(y,\bar{y}|x) = \mathsf{d}A(y,\bar{y}|x) - \Big(\omega^{\alpha\beta}y_\alpha\partial_\beta + \overline{\omega}^{\alpha'\beta'}\bar{y}_{\alpha'}\bar{\partial}_{\beta'}\Big)A(y,\bar{y}|x)$$

Conserved currents in AdS_4

Rank-two unfolded equations = current equations

$$D_2{}^{tw}\,\mathcal{J}(y,\bar{y}|x) = 0$$

OG, M.Vasiliev (2003)

Rank-r twisted adjoint derivative

$$D_r{}^{tw} = \mathrm{d} - \omega^{\alpha\beta} y_{j\alpha}\partial^j_\beta - \overline{\omega}^{\alpha'\beta'}\bar{y}_{j\alpha'}\bar{\partial}^j{}_{\beta'} + e^{\alpha\alpha'}\Big(y^i_\alpha \bar{y}^j_{\alpha'} + \partial^i_\alpha \bar{\partial}^j{}_{\alpha'}\Big)\delta_{ij}, \quad i,j = 1,\dots,r$$

Three-form

$$\Omega(\mathcal{J}) \;=\; e^\alpha{}_{\alpha'} \wedge e^{\beta\alpha'} \wedge e_\beta{}^{\beta'} (\partial^1_\alpha - \partial^2_\alpha)(\bar{\partial}^1_{\beta'} - \bar{\partial}^2_{\beta'})\; \mathcal{J}(y,\bar{y}|x)\Big|_{y=\bar{y}=0}$$

is closed by virtue of current equations

Conserved currents= bilinears of solutions $C_{1,2}$ **of rank-one unfolded equations**

$$\mathcal{J}_\eta(y\,,\bar{y}|x) = \eta\, C_1(y_1,\bar{y}_1|x) C_2(y_2,\bar{y}_2|x)$$

that solve the current equations.

Current parameters η **are differential operators commuting with** D_2^{tw}.

$\mathcal{J}_\eta(y\,,\bar{y}|x)$ **define bilinear conserved charges** $Q_\eta = \int \Omega(\mathcal{J}_\eta)$

Current deformation

Schematically for the flat connection $D = \mathrm{d} + w$

$$\begin{cases} D\omega + L(C, \overline{C}, w) = 0 \\ DC = 0 \\ D_2\mathcal{J} = 0 \end{cases} \Rightarrow \begin{cases} D\omega + L(C, \overline{C}, w) + G(w, \mathcal{J}) = 0 \\ DC + F(w, \mathcal{J}) = 0 \\ D_2\mathcal{J} = 0 \end{cases}$$

Deformed equations in AdS_4 **Sector zero-forms** O.G.,M.Vasiliev (2012)

$$\begin{cases} D^{tw}C + \left(e^{\mu\nu'} y_\mu F^j \bar{\partial}_{j\nu'}\mathcal{J} + e^{\mu\nu'} \bar{y}_{\nu'} \overline{F}^j \partial_{j\nu}\mathcal{I} \right)\Big|_{y=\bar{y}=0} = 0 \\ D_2^{tw}\mathcal{J} = 0 \,, \qquad D_2^{tw}\mathcal{I} = 0 \end{cases}$$

$$\mathcal{N}_\pm = y^\alpha \partial_{\pm\alpha} \,, \qquad \overline{\mathcal{N}_\pm} = \overline{\mathcal{N}}_\pm \,, \qquad F^\pm = \overline{F^\pm} , \qquad \partial_\pm \sim \frac{\partial}{\partial y_1} \pm \frac{\partial}{\partial y_2}$$

$$F^\pm = \frac{\partial}{\partial \mathcal{N}_\pm} \left(\sum_{2m \geq n \geq 0} a_{n,m} \left(\mathcal{N}_+\right)^n \left(\mathcal{N}_-\right)^{2m-n} \sum_{k\geq 0} \frac{\left(\overline{\mathcal{N}}_+ \mathcal{N}_- + \overline{\mathcal{N}}_- \mathcal{N}_+\right)^k}{k!(k+2m+1)!} \right)$$

$a_{n,m}$ **: arbitrary coefficients,**

$|a_{n,m}|$ **reflect freedom in normalization of currents of different spins.**

Phases of $a_{n,m}$ **can be understood as resulting from electric-magnetic-like duality transformations for different spins.**

In terms of two-component spinors $\mathfrak{u}(2,2)$ connections are

$$h^{\alpha\alpha'}\,,\ \omega_\alpha{}^\beta\,,\ \overline{\omega}_{\alpha'}{}^{\beta'}\,,\ f_{\alpha\alpha'}\,.$$

$\mathfrak{u}(2,2)$ flatness conditions lead to zero curvature conditions

$$\begin{aligned}
R^{\alpha\beta'} &= \mathrm{d}h^{\alpha\beta'} - \omega_\gamma{}^\alpha \wedge h^{\gamma\beta'} - \overline{\omega}_{\gamma'}{}^{\beta'} \wedge h^{\alpha\gamma'} = 0\\
R_{\alpha\beta'} &= \mathrm{d}f_{\alpha\beta'} + \omega_\alpha{}^\gamma \wedge f_{\gamma\beta'} + \overline{\omega}_{\beta'}{}^{\gamma'} \wedge f_{\alpha\gamma'} = 0\\
R_\alpha{}^\beta &= \mathrm{d}\omega_\alpha{}^\beta + \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta - f_{\alpha\gamma'} \wedge h^{\gamma'\beta} = 0\\
\overline{R}_{\alpha'}{}^{\beta'} &= \mathrm{d}\overline{\omega}_{\alpha'}{}^{\beta'} + \overline{\omega}_{\alpha'}{}^{\gamma'} \wedge \overline{\omega}_{\gamma'}{}^{\beta'} - f_{\gamma\alpha'} \wedge h^{\gamma\beta'} = 0
\end{aligned}$$

Traceless parts $\omega^L{}_\alpha{}^\beta$ and $\overline{\omega}^L{}_{\alpha'}{}^{\beta'}$ of $\omega_\alpha{}^\beta$ and $\overline{\omega}_{\alpha'}{}^{\beta'}$ describe the Lorentz connection. Traces are associated with the gauge fields

$$b = \frac{1}{2}\Big(\omega_\alpha{}^\alpha + \overline{\omega}_{\alpha'}{}^{\alpha'}\Big) \text{ and } \widetilde{b} = \frac{1}{2}\Big(\omega_\alpha{}^\alpha - \overline{\omega}_{\alpha'}{}^{\alpha'}\Big)\,.$$

$\mathfrak{su}(2,2)$ connections are $h^{\alpha\alpha'}\,,\ \omega^{L\beta}_\alpha\,,\ \overline{\omega}^L_{\alpha'}{}^{\beta'}\,,\ f_{\alpha\alpha'}\,, b;\qquad \widetilde{b} = 0$

AdS_4 geometry is described by the Lorentz connections and vierbein $e^{\alpha\alpha'}$ of $\mathfrak{sp}(4,\mathbb{R}) \subset \mathfrak{u}(2,2)$ via the substitution

$$h^{\alpha\alpha'} = \lambda e^{\alpha\alpha'}\,,\qquad f_{\alpha\alpha'} = \lambda e_{\alpha\alpha'}\,,\qquad b = \widetilde{b} = 0\,.$$

Rank -r unfolded equations

$$D_{r\mathfrak{u}}^{tw} C(y,\bar{y}|x) = 0$$

$$D_{r\mathfrak{u}}^{tw} = \mathrm{d} - \omega^{L\alpha\beta} y_{j\alpha}\partial^j_\beta - \overline{\omega}^{L\alpha'\beta'}\bar{y}_{j\alpha'}\bar{\partial}^j_{\beta'} + f_{\alpha\alpha'} y_i{}^\alpha \bar{y}_j{}^{\alpha'}\delta^{ij} + h^{\alpha\alpha'}\partial^i_\alpha\bar{\partial}^j_{\alpha'}\delta_{ij} + \\ + b\mathcal{D}_r + \widetilde{b}\mathcal{H}_r \qquad (i,j = 1,\dots r)$$

describe $\mathfrak{u}(2,2)$ **invariant** $4d$ **massless fields, while**

$$D_{r\mathfrak{su}}^{tw} C(y,\bar{y}|x) = 0\,, \qquad D_{r\mathfrak{su}}^{tw} = D_{r\mathfrak{u}}^{tw}\Big|_{\widetilde{b}=0}$$

describe $\mathfrak{su}(2,2)$ **invariant** $4d$ **massless fields.**

$\mathcal{H}_r = \frac{1}{2}\Big(y_j{}^\alpha\partial^j{}_\alpha - \bar{y}_j{}^{\alpha'}\bar{\partial}^j{}_{\alpha'}\Big)$ **rank-r helicity operator**

$\mathcal{D}_r = r + \frac{1}{2}\Big(y_j{}^\alpha\partial^j{}_\alpha + \bar{y}_j{}^{\alpha'}\bar{\partial}^j{}_{\alpha'}\Big)$ **rank-r dilatation operator** $j = 1,\dots,r$

$C(y,\bar{y}|x)$ **- generalized Weyl tensors.**

Conformal dimension Δ = eigenvalue of $\mathcal{D}_r$

Spin-s rank-r primary field $\Delta = r + s$

Conformal invariance of the deformation

Conformal deformed equations

$$\begin{cases} D_{1\,\mathfrak{su}}^{\;tw} C + \left(h^{\mu\nu'} y_\mu F^j \bar{\partial}_{j\nu'} \mathcal{J} + h^{\mu\nu'} \bar{y}_{\nu'} \overline{F}^j \partial_{j\nu} \mathcal{I} \right) \Big|_{y=\bar{y}=0} = 0\,, \\ D_{2\,\mathfrak{su}}^{\;tw} \mathcal{J} = 0\,, \qquad D_{2\,\mathfrak{su}}^{\;tw} \mathcal{I} = 0 \end{cases}$$

Consistency follows from the properties of the gluing operators.

Cancellation of the F-dependent part of the $b\,h$ term

$$\begin{aligned} b h^{\mu\nu'} y_\mu \Big\{ \underline{1} + \frac{1}{2} \Big(3 + N^{\pm} \frac{\partial}{\partial N^{\pm}} + \overline{N}^{\pm} \frac{\partial}{\partial \overline{N}^{\pm}} \Big) \\ - \frac{1}{2} \Big(5 + N^{\pm} \frac{\partial}{\partial N^{\pm}} + \overline{N}^{\pm} \frac{\partial}{\partial \overline{N}^{\pm}} \Big) \Big\} F^j \bar{\partial}_{j\nu'} \mathcal{J} \Big|_{y^{\pm}=\bar{y}^{\pm}=0} &= 0 \end{aligned}$$

First term that results from $dh^{\mu\nu'}$ via flatness conditions, accounting for the conformal dimension of the frame field $h^{\mu\nu'}$, compensates the difference between the rank-one and rank-two vacuum contributions to the conformal dimensions.

This proves consistency of the conformal deformation and hence its conformal invariance.

Inconsistency of the $\mathfrak{u}(2,2)$-extension

$$\begin{cases} D_{1\mathfrak{u}}^{tw} C + \left(h^{\mu\nu'} y_\mu F^j \bar{\partial}_{j\nu'} \mathcal{J} + h^{\mu\nu'} \bar{y}_{\nu'} \overline{F}^j \partial_{j\nu} \mathcal{I} \right) \Big|_{y=\bar{y}=0} = 0\,, \\ D_{2\mathfrak{u}}^{tw} \mathcal{J} = 0\,, \qquad D_{2\mathfrak{u}}^{tw} \mathcal{I} = 0 \end{cases}$$

Consistency demands

$$\widetilde{b} h^{\mu\nu'} y_\mu \Big\{ \Big(1 + N^\pm \frac{\partial}{\partial N^\pm} - \overline{N}^\pm \frac{\partial}{\partial \overline{N}^\pm} \Big) - \Big(-1 + N^\pm \frac{\partial}{\partial N^\pm} - \overline{N}^\pm \frac{\partial}{\partial \overline{N}^\pm} \Big) \Big\} F^j \bar{\partial}_{j\nu'} \mathcal{J} \Big|_{y^\pm=\bar{y}^\pm=0} = 0$$

However it equals to $2\widetilde{b} h^{\mu\nu'} y_\mu F^j \bar{\partial}_{j\nu'} \mathcal{J} \Big|_{y^\pm=\bar{y}^\pm=0} \neq 0$

Vacuum contributions in the first and the second terms do not cancel because the helicity operator counts degree of y **minus degree of** $\bar{y}$ **while the gluing operator** $\sim y \frac{\partial}{\partial \bar{y}^\pm}$

Compensation of the non-zero term by adding some $\widetilde{b} G \widetilde{\mathcal{J}} \Big|_{y^\pm=\bar{y}^\pm=0}$?

Consistency demands

$$\widetilde{b} F^j \bar{\partial}_{j\nu'} \mathcal{J} \Big|_{y^\pm=\bar{y}^\pm=0} = \widetilde{b} D_{1\mathfrak{u}}^{tw} G \widetilde{\mathcal{J}} \Big|_{y^\pm=\bar{y}^\pm=0} \equiv \widetilde{b} D_{1\mathfrak{su}}^{tw} G \widetilde{\mathcal{J}} \Big|_{y^\pm=\bar{y}^\pm=0}$$

$\Rightarrow$ **Original deformation is a full** $D_{\mathfrak{su}}^{tw}$**-differential**

$\Rightarrow$ **Original conformal current interaction is trivial, which is not true**

Conclusion

$$\mathfrak{sp}(4) \subset \mathfrak{su}(2,2) \subset \mathfrak{u}(2,2) \subset \mathfrak{sp}(8)$$

The deformation remains consistent for the $\mathfrak{su}(2,2)$ extension of $\mathfrak{sp}(4)$

but not beyond

The interactions preserve conformal symmetry but not $\mathfrak{u}(2,2) \subset \mathfrak{sp}(8)$.