Classical conformal blocks via AdS/CFT correspondence

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Outline

- Classical conformal blocks as geodesic lengths

Fitzpatrick, Kaplan, Walters' 2014
Asplund, Bernamoti, Galli, Hartman' 2014
Hijano, Kraus, Snively' 2015

- AGT combinatorial realization (instead of Zamolodchikov's recursion)

Alday, Gaiotto, Tachikawa' 2010
Alba, Fateev, Litvinov, Tarnopolsky' 2010

- The general consideration of geodesic motions in the bulk
- Five-point configurations: explicit results
- Conclusions and outlooks

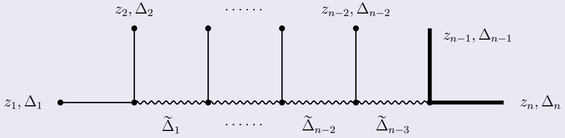
n-point classical conformal block

In any CFT_2 a correlation function of $V_{\Delta_i}(z_i)$ can be decomposed into conformal blocks

$$\mathcal{F}(z_1, ..., z_n|\Delta_1, ..., \Delta_n; \tilde{\Delta}_1, ..., \tilde{\Delta}_{n-3}; c)$$

which are conveniently depicted as

Pant decomposition



There exist many evidences that in the semiclassical limit $c \to \infty$ the conformal blocks must exponentiate as

$$\lim_{c\to\infty} \mathcal{F}(z_1, ..., z_n|\Delta_1, ..., \Delta_n; \tilde{\Delta}_1, ..., \tilde{\Delta}_{n-3}; c) \sim \exp\left\{cf(z_1, ..., z_n|\epsilon_1, ..., \epsilon_n; \tilde{\epsilon}_1, ..., \tilde{\epsilon}_{n-3})\right\}$$

where $\epsilon_k = \frac{\Delta_k}{c}$ and $\tilde{\epsilon}_k = \frac{\tilde{\Delta}_k}{c}$ are called *classical dimensions* and $f(z|\epsilon, \tilde{\epsilon})$ is the *classical* conformal block representing our main interest.

Heavy-light conformal blocks

Different classical limits of the conformal blocks depend on the behavior of the classical dimensions ϵ_i and $\tilde{\epsilon}_i$.

- If $\epsilon, \tilde{\epsilon}$ remain finite in the semiclassical limit, the corresponding field is called heavy.
- If $\epsilon, \tilde{\epsilon}$ are vanishing in the semiclassical limit, the corresponding field is called light.
- All fields are light — *global* $sl(2)$ conformal block.
- All fields are heavy — *proper* classical block.
- Heavy-light classical blocks can be considered as an interpolation between these two extreme regimes.

Heavy-light blocks (Fitzpatrick, Kaplan, Walters' 2014)

The classical conformal dimensions of two fields $\epsilon_{n-1} = \epsilon_n$ are heavy.

It is instructive to introduce a scale factor δ that we call *a lightness parameter*. Schematically, provided that all except two dimensions are rescaled as $\epsilon \to \delta\epsilon$ and $\tilde{\epsilon} \to \delta\tilde{\epsilon}$ there appear a series expansion

$$f(z|\epsilon, \tilde{\epsilon}) = f_{\delta}(z|\epsilon, \tilde{\epsilon})\,\delta + f_{\delta^2}(z|\epsilon, \tilde{\epsilon})\,\delta^2 + \ldots .$$

The leading contribution $f_{\delta}(z)$ yields the heavy-light conformal block, while taking into account sub-leading contributions approximate the proper conformal block on the left hand side.

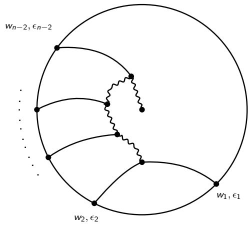
The AdS/CFT correspondence

The heavy operators with equal conformal dimensions $\epsilon_n = \epsilon_{n-1} \equiv \epsilon_h$ produce an asymptotically AdS_3 geometry identified either with an angular deficit or BTZ black hole geometry parameterized by

$$\alpha = \sqrt{1 - 4\epsilon_h}$$

The metric reads

$$ds^2 = \frac{\alpha^2}{\cos^2\rho}\Big(- dt^2 + \sin^2\rho d\phi^2 + \frac{1}{\alpha^2} d\rho^2 \Big)$$



Here

- $\alpha^2 < 0$ for an angular deficit
- $\alpha^2 > 0$ for the BTZ black hole

The light fields are realized via particular graph of worldlines of $n-3$ classical point probes propagating in the background geometry formed by the two boundary heavy fields. Points w_i are boundary attachments of the light operators. The lightness parameter δ measures a backreaction of the background on a probe.

The identification

$$S_{cl}^{bulk} = z^{\gamma} f_{\delta}(z|\epsilon, \tilde{\epsilon}) \;, \qquad S_{cl}^{bulk} = \sum_{i=1}^{n-2} \epsilon_i \, L_i + \sum_{i=1}^{n-3} \tilde{\epsilon}_i \, \tilde{L}_i \;,$$

and L_i and $\tilde{L}_i$ are lengths of different geodesic segments on a fixed time slice.

AGT representation

Using $SL(2)$ invariance we fix three points $z_1 = 0$, $z_{n-1} = 1$, $z_n = \infty$, and replace

$$z_{i+1} = q_i q_{i+1} \dots q_{n-3} \quad \text{for} \quad 1 \leq i \leq n-3$$

The conformal block is given by the following series expansion

$$\mathcal{F}(q|\Delta, \tilde{\Delta}, c) = 1 + \sum_{k_1, \dots k_{n-3}} q_1^{k_1} q_2^{k_2} \dots q_{n-3}^{k_{n-3}} \, \mathcal{F}_k(\Delta, \tilde{\Delta}, c)$$

Using the standard Liouville parametrization,

$$\Delta_i = \frac{Q^2}{4} - P_i^2 \,, \qquad \tilde{\Delta}_j = \frac{Q^2}{4} - \tilde{P}_j^2 \,, \qquad c = 1 + 6Q^2 \,, \qquad Q = b + \frac{1}{b} \,,$$

the AGT representation of the n-point conformal block is given as

$$\mathcal{F}(q|\Delta, \tilde{\Delta}, c) = \prod_{r=1}^{n-3} \prod_{s=r}^{n-3} (1 - q_r \dots q_s)^{2(P_{r+1} - \frac{Q}{2})(P_{s+2} + \frac{Q}{2})} \, \mathcal{Z}(q|\Delta, \tilde{\Delta}, c),$$

where

$$\mathcal{Z}(q|\Delta, \tilde{\Delta}, c) = 1 + \sum_{k_1, \dots k_{n-3}} q_1^{k_1} q_2^{k_2} \dots q_{n-3}^{k_{n-3}} \, \mathcal{Z}_{k_1, \dots k_{n-3}}(\Delta, \tilde{\Delta}, c)$$

The diagrammatic coefficients

The Nekrasov functions

$$\mathcal{Z}_{k_1,\dots k_{n-3}} = \sum_{\vec{\lambda}_1,\dots,\vec{\lambda}_{n-3}} \frac{Z(P_2|P_1,\varnothing;\tilde{P}_1,\vec{\lambda}_1)Z(P_3|\tilde{P}_1,\vec{\lambda}_1;\tilde{P}_2,\vec{\lambda}_2)\cdots Z(P_{n-1}|\tilde{P}_{n-3},\vec{\lambda}_{n-3};P_n,\varnothing)}{Z(\frac{Q}{2}|\tilde{P}_1,\vec{\lambda}_1;\tilde{P}_1,\vec{\lambda}_1)\cdots Z(\frac{Q}{2}|\tilde{P}_{n-3},\vec{\lambda}_{n-3};\tilde{P}_{n-3},\vec{\lambda}_{n-3})}$$

Here, the sum goes over $(n-3)$ pairs of Young tableaux $\vec{\lambda}_j = (\lambda_j^{(1)},\lambda_j^{(2)})$ with the total number of cells $|\vec{\lambda}_j| \equiv |\lambda_j^{(1)}| + |\lambda_j^{(2)}| = k_j$. The explicit form of functions Z reads

$$Z(P''|P',\vec{\mu};P,\vec{\lambda}) =$$

$$\prod_{i,j=1}^{2}\prod_{s\in\lambda_i}\left(P'' - E_{\lambda_i,\mu_j}\big((-1)^j P' - (-1)^i P\big|s\big) + \frac{Q}{2}\right)\times$$

$$\times\prod_{t\in\mu_j}\left(P'' + E_{\mu_j,\lambda_i}\big((-1)^i P - (-1)^j P'\big|t\big) - \frac{Q}{2}\right)$$

where

$$E_{\lambda,\mu}\big(x\big|s\big) = x - b\, l_\mu(s) + b^{-1}(a_\lambda(s)+1)$$

For a cell $s=(m,n)$ such that m and n label a respective row and a column, the arm-length function $a_\lambda(s) = (\lambda)_m - n$ and the leg-length function $l_\lambda(s) = (\lambda)_n^T - m$, where $(\lambda)_m$ is the length of m-th row of the Young tableau λ, and $(\lambda)_n^T$ the height of the n-th column, where T stands for a matrix transposition.

The five-point classical conformal block

$$\mathcal{F}(q_1,q_2) = (1-q_1)^{2(P_2-\frac{Q}{2})(P_3+\frac{Q}{2})}(1-q_1q_2)^{2(P_2-\frac{Q}{2})(P_4+\frac{Q}{2})}(1-q_2)^{2(P_3-\frac{Q}{2})(P_4+\frac{Q}{2})}\mathcal{Z}(q_1,q_2),$$

where

$$\mathcal{Z}(q_1,q_2) = 1 + \sum_{k_1,k_2} q_1^{k_1} q_2^{k_2}\, \mathcal{Z}_{k_1,k_2},$$

and

$$\mathcal{Z}_{k_1,k_2} = \sum_{\vec{\lambda}_1,\vec{\lambda}_2}^{|\vec{\lambda}_{1,2}|=k_{1,2}} \frac{Z(P_2|P_1,\varnothing;\tilde{P}_1,\vec{\lambda}_1)Z(P_3|\tilde{P}_1,\vec{\lambda}_1;\tilde{P}_2,\vec{\lambda}_2)Z(P_4|\tilde{P}_2,\vec{\lambda}_2;P_5,\varnothing)}{Z(\frac{Q}{2}|\tilde{P}_1,\vec{\lambda}_1;\tilde{P}_1,\vec{\lambda}_1)Z(\frac{Q}{2}|\tilde{P}_2,\vec{\lambda}_2;\tilde{P}_2,\vec{\lambda}_2)}\,,$$

where on the lower levels the pairs of Young tableaux $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ with the total number of cells $l = |\vec{\lambda}|$ are

$$\begin{aligned}
l=0: \quad & \{(\varnothing,\varnothing)\}\\
l=1: \quad & \{(\varnothing,\square),(\square,\varnothing)\}\\
l=2: \quad & \{(\varnothing,\square\square),(\varnothing,\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\square,\square),(\square\square,\varnothing),(\begin{smallmatrix}\square\\\square\end{smallmatrix},\varnothing)\}\\
l=3: \quad & \{(\varnothing,\square\square\square),(\varnothing,\begin{smallmatrix}\square\square\\\square\end{smallmatrix}),(\varnothing,\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}),(\square,\square\square),(\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}),\\
& (\square\square,\square),(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square),(\square\square\square,\varnothing),(\begin{smallmatrix}\square\square\\\square\end{smallmatrix},\varnothing),(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\varnothing)\}\,.
\end{aligned}$$

In what follows 5-pt conformal blocks are with dimensions $P_4 = P_5$, $P_1 = P_2$, $\tilde{P}_1 = \tilde{P}_2$.

We find

$$\mathcal{Z}(q_1, q_2|t) = 1 + (1 + b^2 - 2bP_3)(1 + b^2 + 2bP_3)(q_1 + q_2)(8b^2)^{-1}t + \mathcal{O}(t^2) \ ,$$

where t^m terms take into account contributions $q_1^{m_1} q_2^{m_2}$ with $m = m_1 + m_2$.

The limit $c \to 0$ can be equivalently understood as $b \to 0$. The classical conformal block

$$\mathcal{F}(q_1, q_2) = e^{-\frac{f(q_1,q_2)}{b^2}} \qquad \text{or} \qquad f(q_1, q_2) = -\lim_{b\to 0} b^2 \ln \mathcal{F}(q_1, q_2)$$

Fields with $P_4 = P_5$ are heavy. Recall that the lightness parameter expansion is given

$$f(q_1, q_2) = f_\delta(q_1, q_2)\delta + f_{\delta^2}(q_1, q_2)\delta^2 + \ldots$$

Now, ϵ_3 (or P_3) is the new deformation parameter

$$f_\delta(q_1, q_2) = f_\delta^{(0)}(q_1, q_2) + \epsilon_3 f_\delta^{(1)}(q_1, q_2) + \epsilon_3^2 f_\delta^{(2)}(q_1, q_2) + \ldots \ .$$

Here, the leading term $f_\delta^{(0)}(q_1, q_2)$ is identified with the 4-pt classical conformal block, while the sub-leading terms perturbatively reconstruct the 5-pt classical conformal block

$$f_\delta^{(0)}(q_1, q_2) = 2\epsilon_1 \ln\left[-\frac{2\sinh[\frac{\alpha \ln[1-q_1q_2]}{2}]}{\alpha q_1 q_2}\right] - \tilde{\epsilon}_1 \ln\left[-\frac{4\tanh[\frac{\alpha \ln[1-q_1q_2]}{4}]}{\alpha q_1 q_2}\right] + \epsilon_1 \ln[1 - q_1 q_2]$$

and

$$f_\delta^{(1)}(q_1, q_2) = \ln\sinh[\frac{\alpha(\ln[1-q_1q_2] - 2\ln[1-q_2])}{2\alpha q_2}] + \ln[1 - q_2]$$

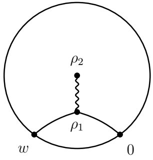
The world-line approach

The semiclassical limit $c \to \infty$. The worldline action $(m \sim \epsilon)$

$$S = \epsilon \int_{\lambda'}^{\lambda''} d\lambda \sqrt{g_{tt}\dot{t}^2 + g_{\phi\phi}\dot{\phi}^2 + g_{\rho\rho}\dot{\rho}^2}\,, \qquad ds^2 = \frac{\alpha^2}{\cos^2\rho}\Big(-dt^2 + \sin^2\rho\, d\phi^2 + \frac{1}{\alpha^2}d\rho^2\Big)$$

It is convenient to impose the normalization condition

$$|\dot{x}| \equiv \sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} = 1 \;: \qquad S = \epsilon \int_{\lambda'}^{\lambda''} d\lambda = \epsilon(\lambda'' - \lambda')\,.$$



Coordinates t and ϕ are cyclic — a constant time disk (ρ,ϕ). Changing variables as $\eta = \cot^2\rho$ and introducing notation $s = \frac{|p_\phi|}{\alpha}$ we find the on-shell action

$$S = \epsilon \ln \frac{\sqrt{\eta}}{\sqrt{1+\eta} + \sqrt{1 - s^2\eta}}\Bigg|_{\eta'}^{\eta''}$$

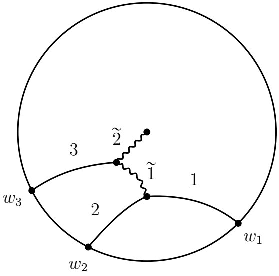
Parameter s is an integration constant that defines a particular form of the geodesic segment.

- The radial line has $s = 0$. For $\rho_1 = \arccos\sin(\alpha w/2)$: $L_{rad} = -\ln\tan\frac{\alpha w}{4}$
- The arc has $s = \cot\frac{\alpha w}{2}$. The length $L_{arc} = \ln\left[\sin\frac{\alpha w}{2}\right] + \ln 2\Lambda$
- The 4-pt block: $f \sim \epsilon_{\tilde{1}} L_{rad} + 2\epsilon_1 L_{arc}$

Five-line configuration

The corresponding particle action reads

$$S = \epsilon_1 L_1 + \epsilon_2 L_2 + \epsilon_3 L_3 + \epsilon_{\tilde{1}} L_{\tilde{1}} + \epsilon_{\tilde{2}} L_{\tilde{2}}$$



Vertex equilibrium equations

- 1st vertex $\left(\tilde{\epsilon}_1 \tilde{p}^1_\mu + \epsilon_1 p^1_\mu + \epsilon_2 p^2_\mu\right)\Big|_{x=x_1} = 0$
- 2nd vertex $\left(\tilde{\epsilon}_1 \tilde{p}^1_\mu + \tilde{\epsilon}_2 \tilde{p}^2_\mu + \epsilon_3 p^3_\mu\right)\Big|_{x=x_2} = 0$

Angular equations

$$\Delta\phi_1 + \Delta\phi_2 = w_2 - w_1\,, \quad \Delta\phi_1 + \Delta\phi_3 + \Delta\tilde{\phi}_1 = w_3 - w_1$$

The complete equation system

Using $p_\rho = g_{\rho\rho}\dot\rho$, $p_\phi = g_{\phi\phi}\dot\phi$ along with the normalization condition, and recalling that the angular momenta are motion constants we find

$$\dot\rho = \cos\rho\,\sqrt{1-s^2\cot^2\rho}\;, \qquad i\alpha\Delta\phi = \ln\frac{\sqrt{1-s^2\cot^2\rho''}-is\sqrt{1+\cot^2\rho''}}{\sqrt{1-s^2\cot^2\rho'}-is\sqrt{1+\cot^2\rho'}}$$

Equations to be solved:

Vertex eqs

$$\epsilon_3\sqrt{1-s_3^2\eta_2}+\tilde\epsilon_1\sqrt{1-\tilde s_1^2\eta_2}=\tilde\epsilon_2\;, \qquad \epsilon_1\sqrt{1-s_1^2\eta_1}+\epsilon_2\sqrt{1-s_2^2\eta_1}=\tilde\epsilon_1\sqrt{1-\tilde s_1^2\eta_1}$$

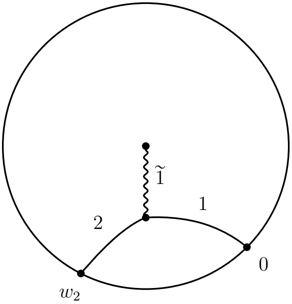
Angular eqs

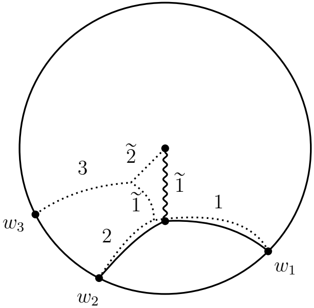
$$e^{i\alpha w_2} = \frac{\left(\sqrt{1-s_1^2\,\eta_1}-is_1\,\sqrt{1+\eta_1}\right)\left(\sqrt{1-s_2^2\,\eta_1}-is_2\,\sqrt{1+\eta_1}\right)}{(1-is_1)(1-is_2)}$$

$$e^{i\alpha w_3} = \frac{\left(\sqrt{1-s_3^2\eta_2}-is_3\sqrt{1+\eta_2}\right)\left(\sqrt{1-\tilde s_1^2\eta_2}-i\tilde s_1\sqrt{1+\eta_2}\right)\left(\sqrt{1-s_1^2\eta_1}-is_1\sqrt{1+\eta_1}\right)}{(1-is_3)\left(\sqrt{1-\tilde s_1^2\eta_1}-i\tilde s_1\sqrt{1+\eta_1}\right)(1-is_1)}$$

- 5-pt case: a complicated higher order algebraic equation
- 4-pt case: an exact solution (Hijano, Kraus, Snively, 2015)

The 5-pt case as a deformation of the 4-pt case





The lines of the resulting five-line configuration are characterized by the deformed angular momenta

$$s_I = b_I + \epsilon_3 c_I + \mathcal{O}(\epsilon_3{}^2) \;, \qquad I = 1, 2, 3, \tilde{1}, \tilde{2} \;,$$

where b_I are the angular momenta of the seed three-line configuration and c_I are corrections. Note that $\tilde{s}_2 = b_{\tilde{2}} = 0$ remain intact, and the seed line $\tilde{1}$ is radial so that $b_{\tilde{1}} = 0$. By convention, b_3 is the seed momentum assigned to line 3. The total action reads

$$S(w_2, w_3) = S_0(w_2) + \epsilon_3 S_1(w_2, w_3) + \mathcal{O}(\epsilon_3{}^2) \;,$$

where $S_0 = S_0(w_2)$ is the action of the three-line configuration, while $S_1(w_2, w_3)$ is a correction.

The total length

We set $\tilde{\epsilon}_1 = \tilde{\epsilon}_2$, $\epsilon_1 = \epsilon_2$. Denote

$$\nu = \epsilon_3/\tilde{\epsilon}_1 \ , \qquad \varkappa = \tilde{\epsilon}_1/\epsilon_1 \ , \qquad \theta_i = \frac{\alpha w_i}{2}$$

The first order solution to the 5-pt configuration reads

$$s_1 = -\cot\theta_2 + \frac{\varkappa}{2\sin\theta_2} - \frac{\nu\varkappa}{2}\cot(2\theta_3 - \theta_2) + \mathcal{O}(\nu^2) \ ,$$

$$s_3 = -\cot(2\theta_3 - \theta_2) + \nu\frac{\cos(2\theta_2 - 4\theta_3) - 2\cos(2\theta_2 - 2\theta_3) - 2\cos 2\theta_3 + 3}{4\sin^3(\theta_2 - 2\theta_3)} + \mathcal{O}(\nu^2) \ ,$$

and

$$s_2 = s_1 - \nu\varkappa s_3 \ , \qquad \tilde{s}_1 = \nu s_3 \ , \qquad \tilde{s}_2 = 0 \ .$$

The final action

$$S(w_2, w_3) = -2\epsilon_1 \ln\sin\theta_2 + \tilde{\epsilon}_1 \ln\tan\frac{\theta_2}{2} - \epsilon_3 \ln\sin(2\theta_3 - \theta_2) + \mathcal{O}(\epsilon_3^2)$$

According to the general prescription the action is related to the conformal block as

$$f_\delta(q_1, q_2) \sim -S(\theta_1, \theta_2)$$

The identification is achieved by the following conformal transformations to the plane

$$\theta_2 = \frac{i\alpha}{2}\ln(1 - q_1 q_2) \ , \qquad \theta_3 = \frac{i\alpha}{2}\ln(1 - q_2)$$

Conclusions & outlooks

Done:

- We have proposed the general identification between the pant decomposition with n legs on the boundary and the corresponding multi-line graph in the bulk.
- We have written down the general system of equations describing the dynamics of probes in the bulk background.
- We have performed explicit computations in the $n = 5$ case establishing the correspondence in the first order in the conformal dimension of one of fields while keeping other dimensions arbitrary.

To be done:

- The n-point configurations explicitly. On the boundary side we can use the monodromic approach.
- The heavy-light classical blocks with arbitrary number of heavy operators.
- The AdS/CFT semiclassical calculations from the first principles.