Invariant Functionals

in

Higher-Spin Theory

M.A.Vasiliev

Lebedev Institute, Moscow

Higher Spin Theories and Holography

Moscow, December 10, 2014

Goal

Despite significant progress in the construction of actions during last thirty years: A.Bengtsson, I.Bengtsson, Brink (1983); Berends, Burgers, van Dam (1984); Fradkin, MV (1987),... Metsaev (2006)... Joung, Tarona (2011) ,...Boulanger, Sundell (2012) ...

construction of the action, generating functional for correlators and BH entropy was lacking

Plan

HS holographic duality from unfolded formulation

Structure of HS equations and Klein operator as de Rham cohomology

Supertrace versus Lagrangians in the extended HS equations

Invariants of the AdS_4 HS theory

Structure of the boundary functional

Conclusion

Unfolded dynamics

Covariant first-order differential equations 1988

$$\mathsf{d}W^{\Omega}(x) = G^{\Omega}(W(x))\,, \qquad G^{\Omega}(W) = \sum_{n=1}^{\infty} f^{\Omega}{}_{\Lambda_1\ldots\Lambda_n} W^{\Lambda_1} \wedge \ldots \wedge W^{\Lambda_n}$$

Geometry is encoded by $G^{\Omega}(W)$: unfolded equations make sense in any space-time

$$\mathsf{d}W^{\Omega}(x) = G^{\Omega}(W(x))\,, \quad x \to X = (x, z)\,, \quad \mathsf{d}_x \to \mathsf{d}_X = \mathsf{d}_x + \mathsf{d}_z\,, \quad \mathsf{d}_z = dz^u \frac{\partial}{\partial z^u}$$

X-dependence is reconstructed in terms of $W(X_0) = W(x_0, z_0)$ at any X_0

Classes of holographically dual models: different G 2012

Nonlinear HS equations

$$\mathcal{W} = (\mathsf{d} + W) + S\,, \qquad W = dx^n W_n\,, \qquad S = dz^\alpha S_\alpha + d\bar{z}^{\dot\alpha}\bar{S}_{\dot\alpha}$$

$$\mathcal{W}\star\mathcal{W} = i(dZ^A dZ_A + dz^\alpha dz_\alpha F(B)\star k\star\kappa + d\bar{z}^{\dot\alpha}d\bar{z}_{\dot\alpha}\bar{F}(B)\star\bar{k}\star\bar{\kappa})\,,$$

$$\mathcal{W}\star B = B\star\mathcal{W}$$

HS star product

$$(f * g)(Z;Y) = \frac{1}{(2\pi)^4}\int d^4U\, d^4V \exp\,[iU_A V^A]\, f(Z+U;Y+U)g(Z-V;Y+V)$$

Manifest gauge invariance

$$\delta\mathcal{W} = [\varepsilon,\mathcal{W}]_\star\,, \qquad \delta B = \varepsilon\star B - B\star\varepsilon\,, \qquad \varepsilon = \varepsilon(Z;Y;K|x)$$

Vacuum solution with $B = 0$

$$\mathcal{W}_0 = \mathcal{W}_0^{1,0} + \mathcal{W}_0^{0,1}\,, \qquad \mathcal{W}_0^{1,0} = dZ^A Z_A\,, \qquad \mathcal{W}_0^{0,1} = W_0(Y|x)$$

Klein operators

Klein operator

$$\kappa = \exp iz_\alpha y^\alpha \,, \qquad \kappa * \kappa = 1$$

$$\kappa * f(z,y) = f(-z,-y) * \kappa$$

For the Weyl star product of z-independent functions

$$(f * g)(y) = \frac{1}{(2\pi)^2} \int d^2u \, d^2v \exp\left[iu_\alpha v^\alpha\right] f(y+u) g(y+v)$$

the Klein operator κ_y is the δ-function

$$\kappa_y = 2\pi\delta^2(y)$$

$$\delta(y) * g(y) = g(-y) * \delta(y) \,, \qquad \kappa_y * \kappa_y = 1 \qquad \sim \mathbf{h}^{-2}$$

The HS Klein operator can be defined as

$$\kappa = \kappa_y * \kappa_z$$

Supertrace

$$str(f(z,y)) = \frac{1}{(2\pi)^2} \int d^2u\, d^2v \exp\left[-iu_\alpha v^\beta\right] f(u,v)$$

$$str(f * g) = str(g * f)$$

For z-independent $f(z,y) = f(y)$

$$str(f(y)) = f(0) \qquad \Longrightarrow \qquad str(\kappa_y) = \infty \sim \delta^2(0)$$

Since supertrace is insensitive to the choice of basis of the star-product algebra

$$str(\kappa) \sim \delta^4(0)$$

In our construction invariant functionals have divergent supertrace.

Klein operators are well-defined with respect to the star product.

HS equations from de Rham cohomology in the twistor space

The star-commutator with $\mathcal{W}_0^{1,0} = dZ^A Z_A$ gives de Rham derivative

$$dZ^A Z_A * f - (-1)^p f * dZ^A Z_A = -2i\mathrm{d}_Z f\,, \qquad \mathrm{d}_Z = dZ^A \frac{\partial}{\partial Z^A}$$

The right-hand side of the HS equations has the structure

$$\mathcal{W} * \mathcal{W} = -i(dZ_A dZ^A + \delta^2(dz)\delta^2(z) * \phi + \delta^2(d\bar{z})\delta^2(\bar{z}) * \bar{\phi})$$

ϕ and $\bar{\phi}$ commute with $\mathcal{W}$.

$\delta^2(dz)\delta^2(z)$ is the De Rham cohomology of d_z.

As a result, the interaction terms form a consistent source that cannot be removed by a local field redefinition.

In the Moyal star product, the equations admit no solution at all.

The HS star-product makes the system solvable in terms of Z, Y.

Extended system

HS equations seemingly leave no room for an invariant action as a space-time p-form built from $\mathcal{W}$ and B since $str(\mathcal{W} * f(B) * \mathcal{W} * g(B)) = 0$.

Zero-forms $str(f(B))$ suffer from divergencies of the supertrace suggested to be regularized by Colombo, Iazeolla, Sezgin and Sundell.

$$- \times - = +$$

The new proposal is to consider Lagrangians that are not of the form $str(L)$ via the following extension of the HS unfolded equations

$$\mathcal{W} * \mathcal{W} = F(c, \mathcal{B}) + \mathcal{L}_i\, c^i\,, \qquad \mathcal{W} * \mathcal{B} = \mathcal{B} * \mathcal{W}\,, \qquad \mathsf{d}\mathcal{L} = 0$$

$\mathcal{W} = \mathsf{d} + W$ and $\mathcal{B}$ are differential forms of odd and even degrees, respectively (both in dx and dZ).

c are x- and dx-independent central elements like $dZ_A dZ^A$, $\delta^2(dz)k * \kappa \dots$

Lagrangians $\mathcal{L}$ are x-dependent space-time differential forms of even degrees valued in the center of the algebra. In this talk: $c_i = I$ $i = 1$

$$\mathcal{L}_i\, c^i = \mathcal{L}\, I$$

Symmetries

The system is consistent because $\mathcal{B}$ commutes with itself and with all c
and $\mathcal{L}$. The gauge transformations are

$$\delta\mathcal{W} = [\mathcal{W}\,,\varepsilon]_*\,, \qquad \delta\mathcal{B} = [\mathcal{B}\,,\varepsilon]_*\,, \qquad \varepsilon = \varepsilon(dx, x, dZ, \ldots)$$

$$\delta\mathcal{B} = \{\mathcal{W}\,,\xi\}\,, \qquad \delta\mathcal{W} = \xi^A \frac{\partial F(c,\mathcal{B})}{\partial \mathcal{B}^A}\,, \qquad \xi = \xi(dx, x, dZ, \ldots)$$

$$\delta\mathcal{L} = d\chi\,, \qquad \delta\mathcal{W} = \chi I\,, \qquad \chi(dx, x)$$

χ- transformation implies equivalence up to exact forms

allowing to choose canonical gauge $\mathcal{W}_I := \pi\mathcal{W} = 0$

π is the projection to I

$$\pi(f(Y,Z|x))) = f(0,0|x)\,, \qquad \pi(f\star g) \neq \pi(g\star f)$$

Gauge transformation preserving canonical gauge

$$\delta\mathcal{L} = \mathsf{d}\chi\,, \qquad \chi = -\pi\Big([\mathcal{W}\,,\varepsilon]_* + \xi^A \frac{\partial F(c,\mathcal{B})}{\partial\mathcal{B}^A}\Big)$$

$\mathcal{L}$ is on-shell closed and gauge invariant modulo exact forms

Actions versus supertrace

Gauge invariant action

$$S = \int_{\Sigma} \mathcal{L}$$

Since $\mathcal{L}$ is closed, it should be integrated over non-contractible cycles

For AdS/CFT the singularity is at infinity

BH invariants (entropies) are associated with $(d-2)$-forms

If the HS algebra possesses a supertrace

$$\mathcal{L} = str(\mathrm{d}\mathcal{W} + \mathcal{W} * \mathcal{W})\Big|_{dZ=0}$$

This suggests that the second term vanishes and hence $\mathcal{L}$ is exact.

Not applicable if $str(\mathcal{W} * \mathcal{W})$ is ill-defined:

$\mathcal{L}$ with well-defined $str(\mathcal{W} * \mathcal{W})$ are exact.

$\mathcal{L}$ with ill-defined $str(\mathcal{W} * \mathcal{W})$ have a chance to be nontrivial.

Invariants of the AdS_4 HS theory

$\mathcal{W}(dZ, dx; Z; Y; \mathcal{K}|x)$ **contains all one- and three-forms in** dZ **and** dx

$\mathcal{B}(dZ, dx; Z; Y; \mathcal{K}|x)$ **contains all zero- and two-forms in** dZ **and** dx

Lagrangians $\mathcal{L}(dx|x)$ **depend on space-time coordinates and differentials.**

Lagrangian relevant to the generating functional of correlators in

AdS_4/CFT_3 **HS holography is a four-form** $\mathcal{L}^4$

Lagrangian relevant to BH entropy is a two-form $\mathcal{L}^2$ **?!**

Extended HS system is

$$i\mathcal{W}*\mathcal{W} = dZ_A dZ^A + \delta^2(dz)F_*(\mathcal{B})k*\kappa + \delta^2(d\bar{z})\bar{F}_*(\mathcal{B})\bar{k}*\bar{\kappa} + G(\mathcal{B})\delta^4(dZ)k*\bar{k}*\kappa*\bar{\kappa} + \mathcal{L}I$$

$$\mathcal{L} = \mathcal{L}^2 + \mathcal{L}^4\,, \qquad G = g + O(\mathcal{B})$$

The g**-dependent term represents de Rham cohomology in the** Z**-space.**

Klein operators give rise to divergent traces and, hence, to nontrivial $\mathcal{L}$

Holography at complex infinity

For manifest conformal invariance introduce

$$y^+_\alpha = \frac{1}{2}(y_\alpha - i\bar{y}_\alpha)\,, \qquad y^-_\alpha = \frac{1}{2}(\bar{y}_\alpha - iy_\alpha)\,, \qquad [y^-_\alpha\,, y^{+\beta}]_* = \delta^\beta_\alpha$$

AdS_4 foliation: $x^n = (\mathbf{x}^a, \mathbf{z})$: $\mathbf{x}^a$ are coordinates of leaves $(a = 0, 1, 2,)$

Poincaré coordinates $\mathbf{z}$ is a foliation parameter

$$W = \frac{i}{\mathbf{z}} d\mathbf{x}^{\alpha\beta} y^-_\alpha y^-_\beta - \frac{d\mathbf{z}}{2\mathbf{z}} y^-_\alpha y^{+\alpha}$$

$$e^{\alpha\dot{\alpha}} = \frac{1}{2\mathbf{z}} dx^{\alpha\dot{\alpha}}\,, \qquad \omega^{\alpha\beta} = -\frac{i}{4\mathbf{z}} d\mathbf{x}^{\alpha\beta}\,, \qquad \bar{\omega}^{\dot{\alpha}\dot{\beta}} = \frac{i}{4\mathbf{z}} d\mathbf{x}^{\dot{\alpha}\dot{\beta}}$$

Vacuum connection can be extended to the complex plane of $\mathbf{z}$ with all components containing $d\bar{\mathbf{z}}$ being zero.

AdS infinity is at $\mathbf{z} = 0$

Generating functional for the boundary correlators

$$S = \frac{1}{2\pi i} \oint_{\mathbf{z}=0} L(\phi)$$

An on-shell closed $(d+1)$-form $L(\phi)$ for a d-dimensional boundary

$$\mathrm{d}L(\phi) = 0\,, \qquad L \neq \mathrm{d}M$$

Structure of the functional

The residue at $\mathbf{z} = 0$ gives the boundary functional of the following structure

$$S_{M^3}(\omega) = \int_{M^3} \mathcal{L}\,, \qquad \mathcal{L} = \frac{1}{2}\omega_{\mathbf{x}}^{\alpha_1\ldots\alpha_{2(s-1)}} e_{\mathbf{x}}^{\alpha_{2s-1}}{}_{\beta} e_{\mathbf{x}}^{\alpha_{2s}\beta}(aC_{\alpha_1\ldots\alpha_{2s}}(\omega)+\bar{a}C_{\dot\alpha_1\ldots\dot\alpha_{2s}}(\omega))$$

Using that

$$aC_{\alpha_1\ldots\alpha_{2s}}(\omega) + \bar{a}C_{\dot\alpha_1\ldots\dot\alpha_{2s}}(\omega) = a_-\mathcal{T}_{-\alpha_1\ldots\alpha_{2s}}(\omega) + a_+\mathcal{T}_{+\dot\alpha_1\ldots\dot\alpha_{2s}}(\omega)$$

$\mathcal{T}_-$ describes local boundary terms

$\mathcal{T}_+$ describes nontrivial correlators via the variation of S_{M_3} over the HS gauge fields $\omega_{\mathbf{x}}^{\alpha_1\ldots\alpha_{2(s-1)}}$

$$\langle J(\mathbf{x}_1)J(\mathbf{x}_2)\ldots\rangle = \frac{\delta^n S_{M^3}(\omega, C(\omega))}{\delta\omega(x_1)\delta\omega(x_2)\ldots}\Big|_{\omega=0}$$

Computation of a_+: work in progress

Conclusions

Formulation of holographic duality at the level of the generating functional from the unfolded formulation of HS equations

The proposed formulation is coordinate-independent and applicable to any boundaries and bulk solutions

Invariant functionals for singular solutions BH entropy(?!) follow from the same construction via the $\mathcal{L}^2$-form

AdS_3/CFT_2: Invariant functional is a two-form: boundary functional is an integral of a one-form: holomorphicity of CFT_2

HS AdS/CFT correspondence

General idea of HS duality   Sundborg (2001), Witten (2001)

AdS_4 HS theory is dual to $3d$ vectorial conformal models

Klebanov, Polyakov (2002), Petkou, Leigh (2005), Sezgin, Sundell (2005); Giombi and Yin (2009);

Maldacena, Zhiboedov (2011,2012); MV (2012); Koch, Jevicki, Jin, Rodrigues (2011-2014);

Giombi, Klebanov; Tseytlin (2013,2014) ...

AdS_3/CFT_2 correspondence   Gaberdiel and Gopakumar (2010)

Analysis of HS holography helps to uncover the origin of AdS/CFT ?!

Despite significant progress in the construction of actions during last thirty years: A.Bengtsson, I.Bengtsson, Brink (1983); Berends, Burgers, van Dam (1984);

Fradkin, MV (1987), ... Boulanger, Sundell (2012) ...

construction of the generating functional for correlators and entropies was lacking

$3d$ conformal equations

Rank-one conformal massless equations Shaynkman, MV (2001)

$$\left(\frac{\partial}{\partial x^{\alpha\beta}} \pm i\frac{\partial^2}{\partial y^\alpha \partial y^\beta}\right)C_j^\pm(y|x) = 0\,, \qquad \alpha,\beta = 1,2\,, \quad j = 1,\dots\mathcal{N}$$

Bosons (fermions) are even (odd) functions of y: $C_i(-y|x) = (-1)^{p_i}C_i(y|x)$

Rank-two equations: conserved currents

$$\left\{\frac{\partial}{\partial x^{\alpha\beta}} - \frac{\partial^2}{\partial y^{(\alpha}\partial u^{\beta)}}\right\}J(u,\,y|x) = 0$$

Gelfond, MV (2003)

$J(u,\,y|x)$: **generalized stress tensor. Rank-two equation is obeyed by**

$$J(u,\,y\,|x) = \sum_{i=1}^{\mathcal{N}} C_i^-(u+y|x)\,C_i^+(y-u|x)$$

Primaries: $3d$ **currents of all integer and half-integer spins**

$$J(u,0|x) = \sum_{2s=0}^{\infty} u^{\alpha_1}\dots u^{\alpha_{2s}}J_{\alpha_1\dots\alpha_{2s}}(x)\,, \qquad \tilde{J}(0,y|x) = \sum_{2s=0}^{\infty} y^{\alpha_1}\dots y^{\alpha_{2s}}\tilde{J}_{\alpha_1\dots\alpha_{2s}}(x)$$

$$J^{asym}(u,y|x) = u_\alpha y^\alpha J^{asym}(x)$$

$$\Delta J_{\alpha_1\dots\alpha_{2s}}(x) = \Delta\tilde{J}_{\alpha_1\dots\alpha_{2s}}(x) = s+1 \qquad \Delta J^{asym}(x) = 2$$

Conservation equation: $\frac{\partial}{\partial x^{\alpha\beta}}\frac{\partial^2}{\partial u_\alpha \partial u_\beta}J(u,0|x) = 0$

Free massless fields in AdS_4

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2\ldots$

Fermions require doubling of fields

$$\omega^{ii}(y,\bar{y}\mid x)\,, \qquad C^{i1-i}(y,\bar{y}\mid x)\,, \qquad i = 0,1\,,$$

$$\bar{\omega}^{ii}(y,\bar{y}\mid x) = \omega^{ii}(\bar{y},y\mid x)\,, \qquad \bar{C}^{i\,1-i}(y,\bar{y}\mid x) = C^{1-i\,i}(\bar{y},y\mid x)$$

$$A(y,\bar{y}\mid x) = i\sum_{n,m=0}^{\infty}\frac{1}{n!m!}y_{\alpha_1}\ldots y_{\alpha_n}\bar{y}_{\dot{\beta}_1}\ldots\bar{y}_{\dot{\beta}_m}A^{\alpha_1\ldots\alpha_n}{}_{,}{}^{\dot{\beta}_1\ldots\dot{\beta}_m}(x)$$

The unfolded system for free massless fields is **MV (1989)**

$\star$ $$R_1^{ii}(y,\bar{y}\mid x) = \eta\,\overline{H}^{\dot{\alpha}\dot{\beta}}\frac{\partial^2}{\partial\bar{y}^{\dot{\alpha}}\partial\bar{y}^{\dot{\beta}}}\;C^{1-i\,i}(0,\bar{y}\mid x) + \bar{\eta}\,H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\;C^{i\,1-i}(y,0\mid x)$$

$\star$ $$\tilde{D}_0 C^{i\,1-i}(y,\bar{y}\mid x) = 0$$

$$R_1(y,\bar{y}\mid x) = D_0^{ad}\omega(y,\bar{y}\mid x) \qquad H^{\alpha\beta} = e^{\alpha}{}_{\dot{\alpha}}\wedge e^{\beta\dot{\alpha}}\,, \quad \overline{H}^{\dot{\alpha}\dot{\beta}} = e_{\alpha}{}^{\dot{\alpha}}\wedge e^{\alpha\dot{\beta}}$$

$$D_0^{ad}\omega = D^L - \lambda e^{\alpha\dot{\beta}}\left(y_{\alpha}\frac{\partial}{\partial\bar{y}^{\dot{\beta}}} + \frac{\partial}{\partial y^{\alpha}}\bar{y}_{\dot{\beta}}\right)\,, \qquad \tilde{D}_0 = D^L + \lambda e^{\alpha\dot{\beta}}\left(y_{\alpha}\bar{y}_{\dot{\beta}} + \frac{\partial^2}{\partial y^{\alpha}\partial\bar{y}^{\dot{\beta}}}\right)$$

$$D^L = d_x - \left(\omega^{\alpha\beta}y_{\alpha}\frac{\partial}{\partial y^{\beta}} + \bar{\omega}^{\dot{\alpha}\dot{\beta}}\bar{y}_{\dot{\alpha}}\frac{\partial}{\partial\bar{y}^{\dot{\beta}}}\right)$$

Field equations at the boundary

Rescaling

$$C(y,\bar{y}|\mathbf{x},\mathbf{z}) = \mathbf{z}\,\mathrm{exp}(y_\alpha \bar{y}^\alpha) T(w,\bar{w}|\mathbf{x},\mathbf{z})\,, \qquad \mathbf{w}^\alpha = \mathbf{z}^{1/2}\mathbf{y}^\alpha\,, \qquad \bar{\mathbf{w}}^\alpha = \mathbf{z}^{1/2}\bar{\mathbf{y}}^\alpha$$

$$W^{jj}(y^\pm|\mathbf{x},\mathbf{z}) = \Omega^{jj}(v^-,w^+|\mathbf{x},\mathbf{z})\,, \qquad \mathbf{v}^\pm = \mathbf{z}^{-1/2}\mathbf{y}^\pm\,, \qquad \mathbf{w}^\pm = \mathbf{z}^{1/2}\mathbf{y}^\pm$$

In the limit $\mathbf{z}\to 0$ free HS equations take the form

$$\left(\mathsf{d}_\mathbf{x} + 2i d\mathbf{x}^{\alpha\beta} v^-_\alpha \frac{\partial}{\partial w^{+\beta}}\right)\Omega^{jj}(v^-,w^+|\mathbf{x},0) = d\mathbf{x}^{\alpha\gamma} d\mathbf{x}^{\beta\gamma}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{+\beta}}\mathcal{T}^{jj}_-(w^+,0\,|\,\mathbf{x},0)$$

$$D_\mathbf{x}\Omega^{jj}_\mathbf{z}(v^-,w^+|\mathbf{x},0)+D_\mathbf{z}\Omega^{jj}_\mathbf{x}(v^-,w^+|\mathbf{x},0) = -\frac{i}{2}d\mathbf{x}^{\alpha\beta}d\mathbf{z}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{+\beta}}\mathcal{T}^{jj}_+(w^+,0\,|\,\mathbf{x},0)$$

$$\left[d_\mathbf{x} - i d\mathbf{x}^{\alpha\beta}\frac{\partial^2}{\partial w^{+\alpha}\partial w^{-\beta}}\right]\mathcal{T}^{j\,1-j}_\pm(w^+,w^-|\mathbf{x},0) = 0$$

$$\mathcal{T}^{jj}_\pm(w^+,w^-|\mathbf{x},0) = \eta T^{j\,1-j}(w^+,w^-\,|\,\mathbf{x},0) \pm \bar{\eta} T^{1-j\,j}(-iw^-,iw^+\,|\,\mathbf{x},0)$$