Higher spins and AdS/CFT dualities

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"Partition functions and Casimir energies
in higher spin AdS_{d+1}/CFT_d" arXiv:1402.5396
with S. Giombi and I. Klebanov

"Higher spins in AdS_5 at one loop: vacuum energy,
boundary conformal anomalies and AdS/CFT" arXiv:1410.3273
"Vectorial AdS_5/CFT_4 duality for spin-one boundary theory" arXiv:1410.4457
with M. Beccaria
"Supergravity one-loop corrections on AdS_7 and AdS_3, higher spins
and AdS/CFT" arXiv:1412.0489
with M. Beccaria and G. Macorini

Motivation: learn about (i) structure of HS theories; (ii) limits of AdS/CFT

AdS$_{d+1}$/CFT$_d$ "light":

free boundary CFT$_d$

(i) "vectorial": e.g. free scalar in fundamental of $U(N)$ or $O(N)$

(ii) "adjoint": e.g. free vector in adjoint of $U(N)$ or $O(N)$

no anomalous dimensions of composite operators

but correlation functions are non-trivial in N

vectorial: bilinear "single-trace" operators $\Phi_i^* \partial ... \partial \Phi_i$

adjoint: multilinear single-trace operators $\mathrm{tr}(\Phi \partial ... \partial \Phi \partial ... \partial \Phi .... \Phi)$

in general, any $d = 3, 4, ...$ and any free conformal field is ok

but restrictons of unitarity, etc.:

$d = 3$: scalars or spinor [Maldacena, Zhiboedov 11]

$d = 4$: scalar, spinor or vector [Stanev 12; Alba, Diab 13]

$d = 6$: scalar,..., tensor – e.g. (2,0) tensor multiplet in susy case

• existence of higher-spin symmetries:
[Vasiliev 04; Boulanger, Ponomarev, Skvortsov, Taronna 13]

• vectorial AdS/CFT:
originally in $d = 3$
free or interacting $O(N)$ fixed point theory [Klebanov, Polyakov 02]

• adjoint AdS/CFT:
e.g. in $d = 4$
$g_{\rm YM} = 0$, fixed N limit of $\mathcal{N} = 4$ SYM – $AdS_5 \times S^5$ string duality:
$\lambda = g^2_{\rm YM} N = 0$ limit of standard $\mathrm{AdS}_5/\mathrm{CFT}_4$

• Dual higher spin theory in AdS:
contains infinite set of (massless and massive) HS fields in AdS
dual to primary operators in boundary CFT

vectorial duality:

• spectrum: Flato-Fronsdal type relation:

$\Phi^*(x)\Phi(x') \to \sum \Phi^*\partial ... \partial\Phi$, e.g., in $d = 4$

$$\{0,0\} \times \{0,0\} = (2;0,0) + \bigoplus_{s=1}^{\infty} (2+s; \tfrac{s}{2}, \tfrac{s}{2})$$

corresponding relation for characters same as

AdS/CFT relation for one-particle partition functions

• correlation functions summarised by interaction vertices in AdS_{d+1}

HS theory: Vasiliev-type theory with AdS vacuum

Aim: learn about HS theory in AdS

• match quantum partition functions on both sides of duality

boundary: $S^1 \times S^{d-1}$, S^d, or Einstein space M^d

bulk: (quotient of) AdS_{d+1}, or asymptotically AdS_{d+1} space

• match Casimir energy on $R \times S^{d-1}$ to vacuum energy in AdS_{d+1}

• match a, c_r conformal anomaly coefficients to AdS_{d+1} counterparts

Some background

• consistent interacting massless higher spin gauge theories:
exist in AdS (or dS) background [Fradkin, Vasiliev 88; Vasiliev 92]
e.g. in bosonic 4d case:
infinite set $s = 1, 2, ..., \infty$ plus $s = 0$ with $m^2 = -2$
action $\sim$ quadratic Fronsdal action plus higher interactions

• vectorial AdS_4/CFT_3: [Klebanov, Polyakov 02]
free 3d complex scalar in fundamental representation of $U(N)$
$L = \partial_m \Phi_i^* \partial_m \Phi_i, \quad i = 1, ..., N$
has tower of conserved higher spin currents
$J_{m_1...m_s} = \Phi_i^* \partial_{(m_1} ... \partial_{m_s)} \Phi_i + ...$
singlet sector – $U(N)$ inv "single-trace" CFT primaries:
$J_s, \quad s = 1, 2, ..., \infty$ with $\Delta = s + 1$ – dual to spin s field in AdS_4
$J_0 = \Phi_i^* \Phi_i$ with $\Delta = 1$ – dual to massive scalar $\Delta(\Delta - 3) = m^2 = -2$
same spectrum of states as in HS theory in AdS_4

HS theory dual to free CFT is non-trivial:

free-theory correlators of J_s should be reproduced by
HS interactions in AdS_4 with coupling $\sim 1/N$
checked for tree 3-point functions [Giombi, Yin; Maldacena, Zhiboedov]

$$S = N \int d^{d+1}x \Big[\sum_s \phi_s(-\nabla^2 + m_s^2)\phi_s + \sum C_{s_1 s_2 s_3}(\nabla)\, \phi_{s_1}\phi_{s_2}\phi_{s_3} + ... \Big]$$

full classical action $S = N\bar{S}$ of HS theory for Vasiliev equations not known

quantum corrections: $\Gamma = N\bar{S} + \Gamma_1 + N^{-1}\Gamma_2 + ...$

one-loop $\Gamma_1(0)$ can be found as quadratic action for ϕ_s is known
[Fronsdal 78; Metsaev 94]

- HS theory "summarizes" correlators of bilinear primaries in free theory
- summing up infinite sets of correlators:

partition functions on non-trivial backgrounds should also match

Other similar $d = 3$ models:

• $O(N)$ model : N real scalars
singlet sector – higher spin conserved currents $\Phi_i \partial_{m_1} ... \partial_{m_s} \Phi_i + ...$
non-trivial for even $s = 2, 4, 6, ...$ plus scalar $\Phi_i \Phi_i$ with $\Delta = 1$
dual to "minimal" HS theory in AdS_4 containing even spins only

• "critical vector model": $L = (\partial \Phi_i)^2 + \lambda (\Phi_i \Phi_i)^2$
IR fixed point seen at large N:
scalar $\Delta = 2 + O(\frac{1}{N})$, J_s bilinears $\Delta = s + 1 + O(\frac{1}{N})$
dual to (non)minimal HS theory with $m^2 = -2$ bulk scalar
with alternative b.c.: $\Delta = 2$

• free or critical $U(N)$ or $O(N)$ fermionic 3d models: [Sezgin, Sundell 02]
dual to "type B" ($s = 1/2$) HS theories:
scalar of "type A" ($s = 0$) theory $\to$ pseudo-scalar

• higher dimensions: vectorial AdS/CFT duality should apply for $d \geqslant 3$

• singlet sector of $U(N)$ or $O(N)$ free scalar CFT$_d$
dual to *non-minimal* $(s = 1, 2, ...)$ or *minimal* $(s = 2, 4, ...)$
HS theory in AdS_{d+1} + scalar with $\Delta = d - 2$, i.e. $m^2 = -2(d-2)$
[Didenko, Skvortsov 13; Giombi, Klebanov, Safdi 14]

• "non-trivial" interacting critical theory only in $d = 3$ or also in $d = 5$?
[Fei, Giombi, Klebanov 14]

• singlet sector may be "dynamically" selected by
gauging $U(N)$ or $O(N)$ symmetry and taking gauge coupling to 0
(e.g. coupling to $k = \infty$ CS in $d = 3$)

• test: compare, e.g., quantum partition functions
of large N CFT on $M^d = S^d,\ S^1 \times S^{d-1}, ...$
and of massless HS theory in AdS_{d+1} with boundary M^d

Example: $M^3 = S^3$ $\qquad Z_{\rm CFT}(S^3) = Z_{\rm HS}(AdS_4)$

free complex $U(N)$ scalar CFT: $\int d^3x \sqrt{g}\, \Phi_i^* (-\nabla^2 + \frac{1}{8} R)\Phi_i$

$$\Gamma_{\rm free} = -\ln Z = N \ln \det(-\nabla^2 + \tfrac{3}{4})$$
$$= N \sum_{n=0}^{\infty} (n+1)^2 \ln[(n+\tfrac{1}{2})(n+\tfrac{3}{2})] = N\big[\tfrac{1}{4}\ln 2 - \tfrac{3}{8\pi^2}\zeta(3)\big]$$

Bulk HS theory: expand near AdS_4 vacuum: $ds^2 = d\rho^2 + \sinh^2\rho\, d\Omega_3$

- vacuum value of (unknown) classical action $S = N\bar{S}$ should match (one-loop) CFT value: remains open problem
- AdS/CFT: all quantum corrections in $\Gamma = N\bar{S} + \Gamma_1 + N^{-1}\Gamma_2 + ...$ should then vanish
- check directly that $\Gamma_1 = 0$

Free action of massless totally symmetric HS fields in AdS_{d+1} is known;
gauge fixing ($\delta\phi_s = \nabla\epsilon_{s-1}$) leads to 1-loop HS partition function:

$$Z_s(AdS_{d+1}) = \Big[\frac{\det\big(-\nabla^2 + \mathrm{m}'^2_{s-1}\big)_{s-1,\perp}}{\det\big(-\nabla^2 + \mathrm{m}^2_s\big)_{s,\perp}}\Big]^{1/2}$$

$$\mathrm{m}^2_s = (s-2)(s+d-2) - s\ , \qquad \mathrm{m}'^2_{s-1} = (s-1)(s+d-2)$$

∇^2 on symmetric transverse traceless tensors (curvature radius $r=1$)
$d=2$, $s \geqslant 2$: [Gaberdiel, Gopakumar, Saha 10]; $d \geqslant 3$: [Gupta, Lal 12]

physical and ghost "mass" terms $\quad \mathrm{m}^2_s = \Delta(\Delta - d) - s$
$\Delta = s + d - 2$ and $\Delta' = s + d - 1$ – dimensions of J_s and ∂J_s
scalar $s=0$: $\quad -\nabla^2 - 2(d-2)$ and no ghost numerator

Compute determinants using AdS heat kernel [Camporesi, Higuchi 92]
spectral ζ-function in non-compact case
$\zeta(z) = \sum_n d_n \lambda_n^{-z} \to \int du\, \mu(u)\, \lambda_u^{-z}$

$$\Gamma_1(AdS_{d+1}) = -\tfrac{1}{2}\zeta(0)\ln(r^2\Lambda^2) - \tfrac{1}{2}\zeta'(0)\ , \qquad \Lambda = (\varepsilon_{\rm UV})^{-1} \to \infty$$

• even $d+1$: log UV divergence $\to$ IR divergence in CFT on S^d
must be absent – UV finiteness: $\sum_s \zeta_s(0) = 0$

• odd $d+1$: $\zeta_s(0) = 0$ but need to show that $\sum_s \zeta'_s(0) = 0$

For $(-\nabla^2 + {\rm m}^2)_{s\perp}$, $\quad {\rm m}^2 = \Delta(\Delta - d) - s$

$$\zeta_{\Delta,s}(z) = c_d\, g_s \int_0^\infty du\, \mu_s(u)\left[u^2 + (\Delta - \tfrac{1}{2}d)^2\right]^{-z}$$

$d=3$:
$c_d = \frac{2^{d-1}}{\pi}\frac{{\rm Vol}(AdS_{d+1})}{{\rm Vol}(S^d)} \to \frac{8}{3\pi}\ , \qquad g_s = 2s+1$
$\mu_s = \frac{\pi u}{16}\left[u^2 + (s+\tfrac{1}{2})^2\right]\tanh \pi u$

UV finiteness of HS theory in AdS_4 vacuum [Giombi, Klebanov 13]

$$\sum_s \zeta_s(0) = \zeta_{1,0}(0) + \sum_{s=1}^{\infty} \left[\zeta_{s+1,s}(0) - \zeta_{s+2,s-1}(0)\right]$$
$$= \tfrac{1}{360} + \tfrac{1}{24} \sum_{s=1}^{\infty} \left(\tfrac{2}{15} - s^2 + 5s^4\right) = 0$$

if regularized with Riemann ζ-function: $\zeta(0) = -\frac{1}{2},\ \zeta(-2n) = 0$

(same if add cutoff $e^{-\epsilon s}$, $\epsilon \to 0$ and drop singular terms)

- this regularization should be required by symmetries of theory
- finiteness is automatic if $\sum_s$ done for fixed UV cutoff Λ and then $\Lambda \to \infty$

can be demonstrated by first summing $\zeta_s(z)$ for arbitrary z

one-loop UV finiteness applies to all bosonic massless HS theories in AdS_{d+1}

Vanishing of finite part of $\Gamma_1(AdS_4)$ [Giombi, Klebanov 13]

$$\Gamma_1 = -\tfrac{1}{2}\zeta'_{1,0}(0) - \tfrac{1}{2}\sum_{s=1}^{\infty}\left[\zeta'_{s+1,s}(0) - \zeta'_{s+2,s-1}(0)\right]$$

$$\zeta'_{\Delta,s}(0) = -\tfrac{1}{3}(2s+1)\int_0^{\Delta-\frac{3}{2}} dv\, v\left[v^2 - (s+\tfrac{1}{2})^2\right]\psi(v+\tfrac{1}{2})$$

HS tower part contribution exactly cancels against scalar part

$$\zeta'_{1,0}(0) = -\frac{1}{1152} - \frac{11}{2880}\ln 2 - \frac{1}{8\pi^2}\zeta(3) + \tfrac{1}{8}\zeta'(-1) + \tfrac{5}{8}\zeta'(-3)$$

1-loop partition function in non-minimal HS theory in AdS_4 vanishes: consistent with no N^0 term in Γ of free $U(N)$ CFT on S^3

In minimal (even spin) HS theory – non-zero one-loop result:

$$\Gamma_{1\ \rm min} = \tfrac{1}{8}\ln 2 - \frac{3}{16\pi^2}\zeta(3)$$

dual to $O(N)$ real scalar CFT where no N^0 correction ?!

$$\Gamma_{\rm free\ O(N)} = N\left[\tfrac{1}{8}\ln 2 - \frac{3}{16\pi^2}\zeta(3)\right]$$

Assume: minimal HS theory coupling $N-1$ not N [Giombi, Klebanov 13]:

$$\Gamma_{0\ \rm min} = (N-1)\bar{S} = (N-1)\big[\tfrac{1}{8}\ln 2 - \tfrac{3}{16\pi^2}\zeta(3)\big]$$

$$\Gamma_{0\ \rm min} + \Gamma_{1\ \rm min} = \Gamma_{\rm free\ O(N)}$$

evidence for $g_{\rm min}^{-1} = N-1$ found also in $M^d = S^1 \times S^d$ case

- same $N-1$ in minimal type B theory (dual to free Majorana fermions)
- in minimal "type C theory" (dual to real N vectors) coupling should be $N-2$ [Beccaria, AT 14]

open questions:

- true meaning of $N \to N-1$

(quantum shift, analogy with CS theory, cf. quantization of HS coupling,...)

- why classical action $\bar{S}(AdS_4) = \frac{1}{8}\ln 2 - \frac{3}{16\pi^2}\zeta(3)$

or there is some interpretational subtlety ?

General d: free scalar CFT on $M^d = S^d \leftrightarrow$ HS theory in AdS_{d+1}

• Vasiliev theory in AdS_{d+1}: totally symm. ϕ_s plus $m^2 = -2(d-2)$ scalar
same spectrum as bilinear primaries in scalar CFT

• similar results about matching of partition functions as in $d = 3$, e.g.,
UV divergences vanish for any d: $\sum_s \zeta_s(0) = 0$

• use of spectral zeta-function
$\zeta_{\Delta,s}(z) = c_d\, g_s \int_0^\infty du\, \mu_s(u) \left[u^2 + (\Delta - \frac{1}{2}d)^2\right]^{-z}$
suggests natural regularization: [Giombi, Klebanov, Safdi 14]
first sum over spins for fixed z and then analytically continue in z;
equivalent to cutoff $e^{-\epsilon \bar{s}}$, $\bar{s} \equiv s + \frac{1}{2}(d-3)$
(same as Riemann zeta-function reg. in $d = 3$ only)

$$\Gamma_1 = -\frac{1}{2}\zeta'_{1,0}(0) - \frac{1}{2}\sum_{s=1}^\infty e^{-\epsilon\bar{s}}\left[\zeta'_{s+1,s}(0) - \zeta'_{s+2,s-1}(0)\right]\Big|_{\epsilon\to 0,\,\text{finite}}$$

Odd d: $AdS_4, AdS_6, AdS_8, ...$

$\Gamma_{\rm CFT}(S^d)$=finite $\sim N$, should be equal to $\Gamma_0(AdS_{d+1}) = N\bar{S}$

• $\Gamma_0 = N\bar{S}$ is finite:
regularized $\mathrm{Vol}(AdS_{d+1}) = \pi^{d/2}\Gamma(-\frac{1}{2}d)$ (drop power IR ∞)

• non-minimal theory ($s = 1, 2, 3, ...$): $\Gamma_1(AdS_{d+1})$ = 0

• minimal theory ($s = 2, 4, 6, ...$): find non-trivial identity (as in $d = 3$)

$$\Gamma_{1\,\rm min}(AdS_{d+1}) = \Gamma_{\rm conf.\,scalar}(S^d)$$

• consistent with AdS/CFT if minimal HS theory coupling is $N - 1$

Even d: $AdS_5, AdS_7, AdS_9, \ldots.$

• $\Gamma_{\rm CFT}(S^d)$ has UV divergence $= -\frac{1}{2} N \zeta(0) \ln(\Lambda^2 r^2)$

$\zeta(0) = B_d(S^d) = -4 {\rm a}_d$, ${\rm a}_d$ = conformal anomaly of scalar in S^d

$B_d \sim \int ({\rm a}_d \mathcal{E}_d + \sum_k {\rm c}_k C_{\ldots} \ldots C_{\ldots}) \rightarrow -2\, {\rm a}_d\, \chi(S^d)$

${\rm a}_4 = \frac{1}{360}$, ${\rm a}_6 = -\frac{1}{4\times 756}$, ${\rm a}_8 = \frac{23}{4\times 113400}, \ldots$

• corresponds to log IR divergence of regularized AdS_{d+1} volume:

$${\rm Vol}(AdS_{d+1}) = \frac{2(-1)^{d/2} \pi^{d/2}}{\Gamma(1+\frac{1}{2}d)} \ln R \,, \qquad R = \varepsilon_{\rm IR}{}^{-1} \rightarrow \infty$$

• $\ln R$ term in classical HS action $\Gamma_0 = N\bar{S} \sim N {\rm Vol}(AdS_{d+1})$

should match $\ln \Lambda = \ln \varepsilon_{\rm UV}^{-1}$ term in $\Gamma_{\rm CFT}(S^d)$: $\varepsilon_{\rm IR} = \varepsilon_{\rm UV} = \varepsilon$

• non-minimal theory: 1-loop correction indeed vanishes $\Gamma_1(AdS_{d+1})$=0

• minimal theory: need again $N \rightarrow N-1$ in classical HS action since

$$\Gamma_{1\,\rm min}(AdS_{d+1}) = \Gamma_{\rm conf.\, scalar}(S^d)$$

Scalar theory in $d = 4$ or symmetric HS theory in AdS_5:

$$\Gamma_1(AdS_5) = -\mathrm{a} \ln \varepsilon_{_{\rm IR}}$$

• in non-minimal theory:

$$\mathrm{a} = -\tfrac{1}{720} \sum_{s=1}^{\infty} s^2(s+1)^2[14s(s+1)+3] = -\tfrac{1}{72}\zeta(-3) - \tfrac{7}{240}\zeta(-5) = 0$$

• in minimal theory:

$$\mathrm{a}_{\rm min} = -\tfrac{1}{720} \sum_{s=2,4,\dots}^{\infty} s^2(s+1)^2[14s(s+1)+3]$$
$$= -\tfrac{1}{9}\zeta(-3) - \tfrac{16}{15}\zeta(-5) = \tfrac{1}{360} = \mathrm{a}_{4\ \rm scalar}$$

agrees with $N \to N-1$ coupling shift

CFT in $M^d = S^1_\beta \times S^{d-1} \leftrightarrow$ HS theory in thermal AdS_{d+1}

[Giombi, Klebanov, AT 14]

• CFT$_d$ in radial quantization: operators in $R^d \to$ states in $R_t \times S^{d-1}$
spectrum of dimensions / energies – in finite $T = \beta^{-1}$ partition function

• dual theory on thermal quotient of $(AdS_{d+1})_\beta$ with boundary $S^1_\beta \times S^{d-1}$

• check matching of thermal partition functions = free energies
also: Casimir energy in $R_t \times S^{d-1} \to$ vacuum energy in AdS_{d+1}

• matching implied by equivalence of the spectra but non-trivial:
(i) singlet constraint in CFT; (ii) summation over spins in AdS
• singlet constraint: $O(N^0)$ term in CFT free energy no longer =0;
one-loop correction in HS theory in $(AdS_{d+1})_\beta$ no longer =0
• HS vacuum energy in AdS_{d+1}: vanishes after sum over spins

Standard relations: $\quad \mathrm{CFT}_d$ in $R_t \times S^{d-1}$

one-particle or canonical partition function

$$\mathcal{Z}(\beta) = \mathrm{tr}\, e^{-\beta H} = \sum_n \mathrm{d}_n\, e^{-\beta\omega_n}$$

"energy" zeta-function

$$\zeta_E(z) = \sum_n \mathrm{d}_n\, \omega_n^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty d\beta\, \beta^{z-1} \mathcal{Z}(\beta)$$

Casimir or vacuum energy

$$E_c = \tfrac{1}{2} \sum_n \mathrm{d}_n\, \omega_n = \tfrac{1}{2}\zeta_E(-1)$$

multi-particle or grand canonical partition function Z and free energy

$$\ln Z(\beta) = \mathrm{tr} \ln \left(1 - e^{-\beta H}\right)^{-1} = -\textstyle\sum_n d_n \ln(1 - e^{-\beta\omega_n})$$

$$F_\beta = -\ln Z(\beta) = -\sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}(m\beta)$$

Free conformal scalar in $S^1_\beta \times S^{d-1}$:

$$\Gamma = -\ln Z = \tfrac{1}{2}\ln\det\Delta_0 \,, \qquad \Delta_0 = -\nabla^2 + \tfrac{d-2}{4(d-1)}R$$

$$\Delta_0 = -\partial_t^2 + \Delta_{S^{d-1}} \,, \qquad \Delta_{S^{d-1}} = -\nabla^2_{S^{d-1}} + \tfrac{1}{4}(d-2)^2$$

spectrum of $\Delta_{S^{d-1}}$

$$\lambda_n = \omega_n^2 \,, \qquad \omega_n = n + \tfrac{1}{2}(d-2) \,, \qquad \mathrm{d}_n = 2[n + \tfrac{1}{2}(d-2)]\tfrac{(n+d-3)!}{(d-2)!n!}$$

eigenvalues of Δ_0: $\qquad \lambda_{k,n} = (\tfrac{2\pi k}{\beta})^2 + \omega_n^2$

$$\zeta_{\Delta_0}(z) = \textstyle\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} \mathrm{d}_n\,(\lambda_{k,n})^{-z}$$

In general:

$$\Gamma = -\zeta_{\Delta_0}(0)\ln\Lambda - \tfrac{1}{2}\zeta'_{\Delta_0}(0) \equiv \widehat{F} = \widehat{F}_\infty + \widehat{F}_c + \widehat{F}_\beta$$

$$\widehat{F}_\infty = a_d \ln\Lambda \,, \qquad \widehat{F}_c = \beta E_c = \tfrac{1}{2}\beta\sum_{n=0}^{\infty}\mathrm{d}_n\,\omega_n$$

$$\widehat{F}_\beta = \beta F(\beta) = \sum_{n=0}^{\infty}\mathrm{d}_n\ln(1-e^{-\beta\omega_n}) = -\sum_{m=1}^{\infty}\tfrac{1}{m}\mathcal{Z}(m\beta)$$

Explicitly: $\widehat{F}_\infty = 0, \quad a_d = 0, \quad \chi(S^1 \times S^{d-1}) = 0$

$$d = \text{odd} \geqslant 3 : \quad \widehat{F} = \widehat{F}_\beta \ ; \qquad d = \text{even} \geqslant 4 : \quad \widehat{F} = \beta E_c + \widehat{F}_\beta$$

$$\widehat{F}_\beta = -\sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}_0(m\beta) \ , \qquad \mathcal{Z}_0(\beta) = \sum_{n=0}^{\infty} \mathrm{d}_n \, e^{-\beta\omega_n} \ = \frac{q^{\frac{1}{2}(d-2)}(1-q^2)}{(1-q)^d}$$

Casimir energy: $E_c = \frac{1}{2}\zeta_E(-1)$

$$E_c = \tfrac{1}{2} \sum_{n=0}^{\infty} \mathrm{d}_n \, (\omega_n)^{-z} \Big|_{z\to -1} = \sum_{n=0}^{\infty} \tfrac{(n+d-3)!}{(d-2)!n!} \left[n + \tfrac{1}{2}(d-2)\right]^{-2z} \Big|_{z\to -1}$$

$$E_c^{(d=\text{odd})} = 0 \ , \qquad E_c^{(d=\text{even})} = \sum_{q=0}^{\frac{1}{2}d-2} \kappa_q \, \zeta(2q+1-d) \ ,$$

$$E_c^{(2)} = \zeta(-1) = -\tfrac{1}{12}, \quad E_c^{(4)} = \tfrac{1}{2}\zeta(-3) = \tfrac{1}{240}, \ \ldots$$

Interpretation of one-particle partition function $\mathcal{Z}_0(\beta)$ in R^d

- counts conf. operators $\mathcal{O}_{m_1 \ldots m_n} = \partial_{m_1} \ldots \partial_{m_n} \Phi$ in R^d modulo $\partial^2 \Phi$=0

[Cardy 91; Kutasov, Larsen 00]

$$\Delta(\Phi) = \tfrac{1}{2}(d-2)\ , \quad \Delta(\mathcal{O}_{m_1 \ldots m_n}) = n + \tfrac{1}{2}(d-2)\ , \quad \mathrm{d}_n = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1}$$

$$\mathcal{Z}_0 = \sum_{\mathcal{O}} q^{\Delta_{\mathcal{O}}} = \sum_{n=1}^{\infty} \mathrm{d}_n\, q^{n+\frac{1}{2}(d-2)} = \frac{q^{\frac{1}{2}(d-2)}(1-q^2)}{(1-q)^d}$$

(e.g. $\prod_{k=1}^{d}(1+\partial_k+\partial_k^2+\ldots)$ gives $(1-q)^{-d}$ and $1-q^2$ is subtr. of e.o.m.)

- also: character of scalar (singleton) representation of $SO(d,2)$ [Dolan 05]

- AdS/CFT: need to count $U(N)$ invariant or singlet operators $\Phi_i^* \partial_{m_1} \ldots \partial_{m_n} \Phi_i$+...
- Singlet constraint can be implemented in path integral by integrating over flat $U(N)$ gauge field with non-trivial holonomy in S^1
- Partition function counting singlet operators turns out to be square of $\mathcal{Z}_0$:

$$\mathcal{Z}_0(\beta) \ \to \ \mathcal{Z}_{\rm U(N)}(\beta) = \left[\mathcal{Z}_0(\beta)\right]^2 = \frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}}\ , \qquad\qquad q = e^{-\beta}$$

- scalar partition function $\sim N$; singlet partition function $\sim N^0$

CFT partition function with singlet constraint

- general relation: [Skagerstam 84]

$$Z = \exp \sum_{m=1}^{\infty} \frac{1}{m} \mathcal{Z}(q^m) \ \rightarrow \ Z_G = \int [dg] \exp \sum_{m=1}^{\infty} \frac{1}{m} \chi(g^m) \, \mathcal{Z}(q^m)$$

χ – character of corresponding rep. of symmetry group G

- Direct derivation from scalar partition function on $S^1_\beta \times S^{d-1}$:

[Sundborg 00; Aharony et al 03; Schnitzer 04]

couple complex $U(N)$ scalars Φ_i to gauge field with const holonomy in S^1

$$\partial_t^2 \rightarrow (\partial_t + A_0)^2, \qquad A_0 = g^{-1}\partial_0 g, \qquad g = \text{diag}(e^{i\frac{\alpha_1}{\beta}t}, ..., e^{i\frac{\alpha_N}{\beta}t})$$

$$Z_{\rm U(N)} = \int \prod_{k=1}^{N} d\alpha_k \ e^{-\widetilde{F}(\alpha,\beta)} \ , \qquad \widetilde{F} = -\sum_{i\neq j}^{N} \ln|\sin\tfrac{\alpha_i-\alpha_j}{2}| + \bar{F}(\alpha,\beta)$$

$$\bar{F} = \ln\det\big[-(\partial_t + A_0)^2 + \Delta_{_{S^{d-1}}}\big] = \sum_{i=1}^{N}\sum_{k,n}^{\infty} {\rm d}_n \ln\Big[\tfrac{(2\pi k+\alpha_i)^2}{\beta^2} + \omega_n^2\Big]$$

$$= -\sum_{m=1}^{\infty} \tfrac{1}{m}\, c_m(\alpha)\, \mathcal{Z}_0(m\beta) \ , \qquad c_m(\alpha) = 2\sum_{i=1}^{N} \cos m\alpha_i$$

Large N limit:

$\{\alpha_i\} \to \rho(\alpha)$;   measure $\sim N^2$, $\bar{F} \sim N$

saddle point $\rho(\alpha) = \frac{1}{2\pi} + \frac{1}{N}\widetilde{\rho}(\alpha)$; integrate over $\widetilde{\rho}$: [Shenker, Yin 11]

$$\widehat{F}_{\rm U(N)} = -\ln Z_{\rm U(N)} = 2N\beta E_c + \widehat{F}_\beta + O(N^{-1})\ ,$$

$$\widehat{F}_\beta = -\sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}_{\rm U(N)}(m\beta)\ , \qquad \mathcal{Z}_{\rm U(N)}(\beta) = \big[\mathcal{Z}_0(\beta)\big]^2$$

• in real scalar $O(N)$ case: [Giombi, Klebanov, AT 14; Jevicki et al 14]

$$\mathcal{Z}_{\rm O(N)}(\beta) = \tfrac{1}{2}\big[\mathcal{Z}_0(\beta)\big]^2 + \tfrac{1}{2}\mathcal{Z}_0(2\beta) = \tfrac{1}{2}\frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}} + \tfrac{1}{2}\frac{q^{d-2}(1+q^2)}{(1-q^2)^{d-1}}$$

$O(N^0)$ terms in CFT free energy should match 1-loop terms
in free energies of corresponding HS theories in AdS_{d+1}

Higher spin partition function in thermal AdS$_{d+1}$ with $S^1 \times S^{d-1}$ bndry

$$Z = \prod_s Z_s = e^{-\widehat{F}(\beta)} \ , \qquad \widehat{F} = \sum_s \widehat{F}^{(s)} \ , \qquad \widehat{F}^{(s)} = -\ln Z_s$$

$$Z_s = \Big(\frac{\det\big[-\nabla^2 + (s-1)(s+d-2)\big]_{s-1,\perp}}{\det\big[-\nabla^2 + (s-2)(s+d-2) - s\big]_{s,\perp}} \Big)^{1/2}$$

$\widehat{F}$ is UV finite as in S^4 bndry case: $\mathrm{a}_{d+1} = 0$ (local property of AdS_{d+1})

$$\widehat{F} = \widehat{F}_c + \widehat{F}_\beta, \qquad \widehat{F}_c = \beta E_c \ , \qquad \widehat{F}_\beta = \beta F(\beta)$$

To compute non-trivial part $\widehat{F}_\beta$:

• Hamiltonian approach [Allen, Davis 83; Gibbons, Perry, Pope 06]
and group theory to determine energy spectrum of spin s in global AdS$_{d+1}$
with reflective boundary conditions [Avis et al; Breitenlohner, Freedman 82]

• path integral approach – heat kernel for H^{d+1} [Camporesi, Higuchi 92]
and method of images – thermal AdS_{d+1} as quotient H^{d+1}/Z
[Gopakumar, Gupta, Lal 11]

Temperature-dependent part of AdS free energy

$$F_\beta^{(s)} = -\sum_{m=1}^{\infty} \tfrac{1}{m} \mathcal{Z}_s(m\beta) \,, \qquad \mathcal{Z}_s(\beta) = \frac{\mathrm{d}_s\, q^{s+d-2} - \mathrm{d}_{s-1}\, q^{s+d-1}}{(1-q)^d}$$

$\mathrm{d}_s = 2[s + \frac{1}{2}(d-2)] \frac{(s+d-3)!}{(d-2)!\, s!}$ – STT tensors in d dimensions

$\mathrm{d}_s\big|_{d=3} = 2s+1, \;\; \mathrm{d}_s\big|_{d=4} = (s+1)^2, \ldots$

From CFT$_d$ side: $\mathcal{Z}_s$ is character of $SO(d,2)$ rep. containing spin s primary of dim $\Delta = s + d - 2$ and its descendants

[Dolan 05; Gibbons, Perry, Pope 06]

- for HS theory with $\Delta = d-2$ scalar with $\mathcal{Z}_0^{(\Delta)} = \frac{q^\Delta}{(1-q)^d}$:

$$F_\beta = \sum_{s=0}^{\infty} F_\beta^{(s)} = -\sum_{m=1}^{\infty} \tfrac{1}{m}\, \mathcal{Z}(m\beta)$$

$$\mathcal{Z}(\beta) = \mathcal{Z}_0^{(d-2)} + \sum_{s=1}^{\infty} \mathcal{Z}_s(\beta) = \frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}}$$

matches N^0 term in singlet-sector free energy of complex $U(N)$ scalar

• Non-trivial consistency check: bulk and boundary have same spectrum

• Interpretation: one-particle partition function as character $\mathcal{Z}_s(q)$ of $SO(d,2)$: matching implied by group-theoretic Flato-Fronsdal type relation

$$\{0,0\} \times \{0,0\} = (d-2;0,0) + \bigoplus_{s=1}^{\infty}(d-2+s;\tfrac{s}{2},\tfrac{s}{2})$$

$$\left[\mathcal{Z}_0(\beta)\right]^2 = \mathcal{Z}_0^{(d-2)}(\beta) + \sum_{s=1}^{\infty}\mathcal{Z}_s(\beta)$$

• For minimal Vasiliev theory in AdS_{d+1}:

$$F_{\beta\ \mathrm{min}} = \sum_{s=0,2,4,..}^{\infty} F_\beta^{(s)} = -\sum_{m=1}^{\infty}\tfrac{1}{m}\mathcal{Z}_{\mathrm{min}}(m\beta)$$

$$\mathcal{Z}_{\mathrm{min}}(\beta) = \mathcal{Z}_0^{(d-2)} + \sum_{s=2,4,...}^{\infty}\mathcal{Z}_s(\beta) = \tfrac{1}{2}\frac{q^{d-2}(1+q)^2}{(1-q)^{2d-2}} + \tfrac{1}{2}\frac{q^{d-2}(1+q^2)}{(1-q^2)^{d-1}}$$

matches order N^0 term in free energy of $O(N)$ singlet-sector CFT
group-theoretic interpretation?

Casimir energy

similar pattern of matching: order N in CFT to match classical HS part

no 1-loop correction in non-minimal case: HS AdS vacuum energy vanishes

$$\zeta_E(z) = \frac{1}{\Gamma(z)} \int_0^\infty d\beta\, \beta^{z-1}\, \mathcal{Z}(\beta)\,, \qquad \mathcal{Z}(\beta) = \frac{e^{-(d-2)\beta}(1+e^{-\beta})^2}{(1-e^{-\beta})^{2d-2}}$$

$$E_c = \tfrac{1}{2}\zeta_E(-1) = \textstyle\sum_{s=0}^\infty E_{c,s} = 0$$

$\mathcal{Z}(\beta) = \mathcal{Z}(-\beta)$ property implies vanishing of $\zeta_E(-1)$ for all d

individual spin contributions:

$$E_{c,s} = \frac{1}{2}\sum_{n=1}^{\infty} \binom{n+d-2}{d-1} \Big[\mathrm{d}_s(n+s+d-3) - \mathrm{d}_{s-1}(n+s+d-2) \Big]$$

$$d=3: \qquad E_{c,s} = \tfrac{1}{8}s^4 - \tfrac{1}{12}s^2 + \tfrac{1}{240}$$

AdS_4: $E_{c,s}$ computed using standard ζ-function in n [Allen, Davis 83]

• $E_{vac} = 0$ in $\mathcal{N} > 4$ extended gauged supergravities from susy sum rules
$\sum_s (-1)^{2s} d(s)\, s^p = 0, \qquad p < \mathcal{N} = 1, ..., 8, \qquad s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$
$E_c = 0$ in $\mathcal{N} > 4$ extended gauged supergravities [Allen, Davis 83]
and also at each KK level of spectrum of 11-d supergravity on S^7
[Gibbons, Nicolai 84; Inami, Yamagishi 84]

• cancellation in purely bosonic HS theory:

$$E_c(AdS_4) = \tfrac{1}{480} + \sum_{s=1}^{\infty} \left(\tfrac{1}{8}s^4 - \tfrac{1}{12}s^2 + \tfrac{1}{240}\right) = 0$$

since $\zeta(0) = -\frac{1}{2}$, $\zeta(-2) = \zeta(-4) = 0$

$$E_c(AdS_5) = -\tfrac{1}{1440} \sum_{s=0}^{\infty} s(s+1)\Big[18s^2(s+1)^2 - 14s(s+1) - 11\Big] = 0$$

• instead of susy here ζ-function regul. (consistent with symmetries):
no need to use special prescription to sum over s in each d:
automatically get zero if sum over spins is done first for finite z in $\zeta_E(z)$

Non-minimal vs minimal HS theory:

odd d: in CFT $E_c = 0$ and in AdS_{d+1} sum overs spins gives
$E_c = 0$ both in non-minimal (all s) and minimal (even s) HS theory
even d: in CFT $E_c \sim N$ and should match classical HS action
1-loop $E_c = 0$ in non-minimal case but $E_c \neq 0$ in minimal HS case:
using $\zeta_E(z)$ find that

$$E_c^{\min} = \sum_{s=0,2,4,\ldots} E_{c,s} = \sum_{n=0}^{\infty} \frac{(n+d-3)!}{(d-2)!n!} \left[n + \tfrac{1}{2}(d-2)\right]^2$$

i.e. same as Casimir energy of single real conformal scalar in $R \times S^{d-1}$
• again consistent with $N \to N-1$ shift of coupling constant
in minimal HS theory dual to $O(N)$ real scalar CFT

• equivalence of scalar Casimir energy in $R \times S^{d-1}$ and minimal HS energy
in AdS$_{d+1}$ requires use of same (zeta-function) regularization of sum over
radial quantum number n on both sides of AdS/CFT duality

Conclusions

• quantum tests of vectorial – higher spin AdS/CFT

• massless HS theories in AdS_{d+1} at one loop:
UV finite partition function; vanishing vac energy; matching free energies

• importance of definition / regularization of sum over infinite set of spins

Questions:
• leading large N term – classical action of Vasiliev theory?
• meaning of $N \to N-1$ shift in minimal HS theory?
• correlation functions:
sum over spins prescription in intermediate channel;
consistency with $N \to N-1$; etc

AdS$_5$/CFT$_4$: mixed $SO(2,4)$ representations

- type A HS theory dual to $U(N)$ or $O(N)$ scalars:

bilinear currents are totally symmetric traceless tensors

- $d \geqslant 4$: conformal fields and dual HS in AdS not only totally symmetric
- $d = 4$: mixed-symmetry reps – $SO(4)$ Young tableau with two rows

lengths $h_1 = j_1 + j_2 = s, \;\; h_2 = j_1 - j_2, \;\; SU(2) \times SU(2)$ weights (j_1, j_2)

conformal fields in $SO(2,4)$ reps. $\quad (\Delta; j_1, j_2)$

$j_1 = j_2$: totally symmetric case

- such mixed-symmetry fields appear in e.g. $d = 4$ free fermion

or free Maxwell vector theory and dual type B and C HS theories in AdS$_5$

and thus also in $\mathcal{N} = 4$ Maxwell multiplet (superdoubleton) theory

- important for understanding (limits of) adjoint AdS/CFT

Aim:

- compute boundary conformal anomalies a and c;

partition function and Casimir energy for generic $(\Delta; j_1, j_2)$ field

- check AdS/CFT in type B and type C theories in AdS$_5$

AdS_5	CFT_4 (singlet sector)
non-minimal type A theory $(2;0,0)+\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N complex scalars : $U(N)$
minimal type A theory $(2;0,0)+\bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N real scalars : $O(N)$
non-minimal type B theory $2\,(3;0,0)+$ $2\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=1}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Dirac fermions : $U(N)$
minimal type B theory $2\,(3;0,0)+$ $\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Majorana fermions : $O(N)$
non-minimal type C theory $2\,(4;0,0)+(4;1,0)_c+$ $2\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N complex Maxwell vectors : $U(N)$
minimal type C theory $2\,(4;0,0)+$ $\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N real Maxwell vectors : $O(N)$

4d conformal anomaly

$$\mathcal{A} = -\mathrm{a}\,\mathcal{E} + \mathrm{c}\,C^2 + \mathrm{g}\,D^2 R$$

Casimir energy on S^3 [Cappelli, Coste 89]

$$E_c = \tfrac{3}{4}\left(\mathrm{a} + \tfrac{1}{2}\mathrm{g}\right)$$

g and E_c both depend on regularization (natural: ζ-function or heat kernel)

$\mathcal{N} \geqslant 3$ supersymmetric case (e.g. $\mathcal{N} = 4$ SYM)

$$\mathcal{N} \geqslant 3 \ \ \text{susy}: \qquad E_c = \tfrac{3}{4}\mathrm{a}\,, \qquad \mathrm{a} = \mathrm{c}\,, \qquad \mathrm{g} = 0$$

• extract 4d conformal anomaly from bulk description:

(cf. "tree-level" 5d derivation of conf. anom. [Henningson, Skenderis 98])

1-loop correction:

$\mathcal{O} = -D^2 + X$ for 5d field ϕ dual to 4d field $(\Delta; j_1, j_2)$ [Metsaev]

$$\mathcal{O} = -D^2 + X\,, \qquad X = \Delta\,(\Delta - 4) - h_1 - |h_2| = (\Delta - 2)^2 - 2\,j_1$$

on asymptotically AdS$_5$ space $ds^2 = z^{-2}\,[dz^2 + g_{\mu\nu}(x,z)\,dx^\mu\,dx^\nu]$

1-loop partition function with Dirichlet-type "+" or Neumann-type "−" b.c.

$$Z^{\pm} = (\det \mathcal{O})_{\pm}^{-1/2}$$

boundary conformal anomaly $\mathcal{A}^{\pm}$ as variation of $Z^{\pm}$:

$\delta \log Z^{\pm} = -\frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, \delta\sigma\, \mathcal{A}^{\pm}, \quad \delta g_{\mu\nu} = 2\, \delta\sigma\, g_{\mu\nu}$

early attempt [Mansfield, Nolland, Ueno 03]: $\mathcal{A}^{+} = (\Delta - 2)\, \bar{\mathcal{A}}$

in general $\mathcal{A} = \mathcal{A}^{-} - \mathcal{A}^{+} = -2\mathcal{A}^{+}$ and $\bar{\mathcal{A}}$ is function of (Δ, j_1, j_2)

now found explicitly in case of S^4 boundary; conjectured for $R_{\mu\nu} = 0$

Partition function on $S^1 \times S^3$ and Casimir energy

one-particle partition functions same as conformal characters [Dolan 05]

"massive" conformal rep. $(\Delta;\, j_1, j_2)$: $\Delta > 2 + j_1 + j_2$

long representation of $SO(2,4)$ – massive AdS$_5$ HS field partition function

$$\widehat{\mathcal{Z}}^{+}(\Delta;\, j_1, j_2) = (2j_1 + 1)(2j_2 + 1)\, \frac{q^{\Delta}}{(1-q)^4}$$

"massless" rep: $\Delta = 2 + j_1 + j_2$ corresponds to conserved current in CFT massless HS gauge field in AdS_5 (subtract ghost in 5d or cons. cond. in 4d)

$$\mathcal{Z}^+(\Delta;\, j_1, j_2) = \widehat{\mathcal{Z}}^+(\Delta;\, j_1, j_2) - \widehat{\mathcal{Z}}^+(\Delta+1;\, j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2})\,,$$

$$\mathcal{Z}^+(\Delta;\, j_1, j_2) = \mathcal{Z}_+(\Delta;\, j_1, j_2) = \frac{q^\Delta}{(1-q)^4}\Big[(2j_1+1)(2j_2+1) - 4\,q\,j_1\,j_2\Big]$$

Casimir energy on S^3

compute from $\mathcal{Z}$:

$$E_c = \tfrac{1}{2}\,(-1)^F \sum_n \mathrm{d}_n\,\omega_n = \tfrac{1}{2}(-1)^F\,\zeta_E(-1)$$

$$\zeta_E(z) = \sum_n \frac{\mathrm{d}_n}{\omega_n^z} = \frac{1}{\Gamma(z)}\int_0^\infty d\beta\,\beta^{z-1}\,\mathcal{Z}(e^{-\beta})$$

$$E_c(\Delta;\, j_1, j_2) = \ E_c^- \ - \ E_c^+ = -2\,E_c^+$$

massive rep:

$$\widehat{E}_c(\Delta;\, j_1, j_2) = -\tfrac{1}{720}(-1)^{2j_1+2j_2}\,(2j_1+1)(2j_2+1)(\Delta-2)$$
$$\times\Big[6\,(\Delta-2)^4 - 20\,(\Delta-2)^2 + 11\Big]$$

massless rep. $\Delta = 2 + j_1 + j_2$

$$E_c(\Delta;\, j_1, j_2) = \widehat{E}_c(\Delta;\, j_1, j_2) - \widehat{E}_c\left(\Delta + 1;\, j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}\right)$$

Conformal anomaly a-coefficient

euclidean AdS_5 with S^4 boundary

$$\log Z^+ = -\tfrac{1}{2} \log \det{}_+ \mathcal{O} = \tfrac{1}{2}\, \zeta'(0) = -4\mathrm{a}^+ \log \mathrm{R} + \ldots$$

$\zeta(z)$ from $\mathbb{H}^5$ heat kernel for “massive” 5d operator $\mathcal{O}$
gives for $\mathrm{a} = -2\mathrm{a}^+$ in massive case

$$\widehat{\mathrm{a}}(\Delta;\, j_1, j_2) = \tfrac{1}{720}(-1)^{2(j_1+j_2)}(2j_1 + 1)(2j_2 + 1)(\Delta - 2) \\ \times \Big[-3(\Delta - 2)^4 + 10\big(j_1^2 + j_2^2 + j_1 + j_2 + \tfrac{1}{2}\big)(\Delta - 2)^2 \\ - 15(j_1 - j_2)^2(j_1 + j_2 + 1)^2 \Big]$$

in massless case:

$$\mathrm{a}(\Delta;\, j_1, j_2) = \widehat{\mathrm{a}}\big(\Delta;\, j_1, j_2\big) - \widehat{\mathrm{a}}\big(\Delta + 1;\, j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}\big)$$

Conformal anomaly c-coefficient

if a is known, to find c compute $\mathrm{c} - \mathrm{a}$ on Ricci flat 4d space: $\mathcal{A} = (\mathrm{c} - \mathrm{a})\mathcal{E}$

for low-spin massive fields $\mathrm{c} = -2\mathrm{c}^+$ [Mansfield et al 03; Ardehali et al 13]

$$\widehat{\mathrm{c}}^+ - \widehat{\mathrm{a}}^+ = -\tfrac{1}{360}\,(-1)^{2\,(j_1+j_2)}(\Delta-2)\,d(j_1,j_2)\,\big[1+f(j_1)+f(j_2)\big]$$

$$d(j_1,j_2) = (2j_1+1)(2j_2+1), \qquad f(j) \equiv j\,(j+1)\;[6j\,(j+1)-7]$$

proposal in general case:

$$\widehat{\mathrm{c}}(\Delta;\,j_1,j_2) = \tfrac{1}{720}\,(-1)^{2(j_1+j_2)}(2j_1+1)(2j_2+1)\,(\Delta-2)$$
$$\times\Big[-6\,(\Delta-2)^4+20\,(\Delta-2)^2+6\,(j_1^4+j_2^4)+20\,j_1^2j_2^2+12\,(j_1^3+j_2^3)$$
$$+20\,(j_1^2j_2+j_1j_2^2)-6\,(j_1^2+j_2^2)+20\,j_1j_2-12\,(j_1+j_2)-8\Big]$$

Thus: E_c, a and c are (5-th order) polynomials in $\Delta-2$, and in j_1, j_2

E_c, a, c for superconformal $SU(2,2|\mathcal{N})$ multiplets

• $\mathcal{N}=1$ superconformal multiplets

$\mathcal{N}=1$ multiplets containing $(\Delta;\, j_1,j_2)$ as lowest dim member

(i) long massive multiplets; (ii) shortened ones

(iia) chiral/anti-chiral; (iib) right-handed/left-handed semi-long (SLII/SLI)

$SO(2,4)$ representation content of massive long $\mathcal{N}=1$ multiplet

$$
\begin{aligned}
[\Delta;\, j_1,j_2]_{\rm long} &= (\Delta;\, j_1,j_2) + \left(\Delta+\tfrac12;\, j_1+\tfrac12,j_2\right) + \left(\Delta+\tfrac12;\, j_1-\tfrac12,j_2\right)\\
&+ \left(\Delta+\tfrac12;\, j_1,j_2+\tfrac12\right) + \left(\Delta+\tfrac12;\, j_1,j_2-\tfrac12\right) + 2\,(\Delta+1;\, j_1,j_2)\\
&+ \left(\Delta+1;\, j_1+\tfrac12,j_2+\tfrac12\right) + \left(\Delta+1;\, j_1+\tfrac12,j_2-\tfrac12\right)\\
&+ \left(\Delta+1;\, j_1-\tfrac12,j_2+\tfrac12\right) + \left(\Delta+1;\, j_1-\tfrac12,j_2-\tfrac12\right)\\
&+ \left(\Delta+\tfrac32;\, j_1,j_2+\tfrac12\right) + \left(\Delta+\tfrac32;\, j_1,j_2-\tfrac12\right)\\
&+ \left(\Delta+\tfrac32;\, j_1-\tfrac12,j_2\right) + \left(\Delta+\tfrac32;\, j_1+\tfrac12,j_2+\tfrac12\right) + (\Delta+2;\, j_1,j_2)
\end{aligned}
$$

$$
\mathrm{a}_{\rm long} = \mathrm{c}_{\rm long} = 0, \qquad E_{c\ \rm long} = -\frac{1}{16}\,(-1)^{2\,(j_1+j_2)}\,(2j_1+1)(2j_2+1)\,(\Delta-1)
$$

E_c not proportional to a: g of the D^2R is non-zero in $\mathcal{N}=1$ case chiral short multiplet:

$$
\begin{aligned}
[\Delta;\, j,0]_{\text{chiral}} &= (\Delta;\, j,0) + (\Delta+\tfrac{1}{2};\, j+\tfrac{1}{2},0) + (\Delta+\tfrac{1}{2};\, j-\tfrac{1}{2},0) + (\Delta+1;\, j,0) \\
\mathrm{a}_{\text{chiral}} &= \tfrac{1}{96}\,(-1)^{2j}\,(2j+1)\,(2\,\Delta-3)\,(-2\,\Delta^2+6\,\Delta+6j^2+6j-3) \\
\mathrm{c}_{\text{chiral}} &= -\tfrac{1}{48}\,(-1)^{2j}\,(2j+1)\,(2\,\Delta-3)\,(\Delta^2-3\,\Delta+j^2+j+1) \\
E_{c\ \text{chiral}} &= -\tfrac{1}{384}\,(-1)^{2j}\,(2j+1)\,(16\Delta^3-72\Delta^2+94\Delta-33)
\end{aligned}
$$

$\mathcal{N}$	ϕ	ψ	V_μ	E_c	a	c
1	–	1	1	$\frac{7}{64}$	$\frac{3}{16}$	$\frac{1}{8}$
2	2	2	1	$\frac{13}{96}$	$\frac{5}{24}$	$\frac{1}{6}$
3, 4	6	4	1	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{4}$

$\mathcal{N} > 1$ superconformal multiplets

Maxwell supermultiplets

$$\mathcal{N} = 3, 4 : \qquad E_c = \tfrac{3}{4}\mathrm{a}\,, \qquad \mathrm{a} = \mathrm{c}\,, \qquad \mathrm{g} = 0$$

$\mathcal{N} = 4$ Maxwell multiplet same as $\mathcal{N} = 4$ superdoubleton of $PSU(2,2|4)$
$\{\mathcal{N} = 4\} = \{\,1, 0\}_c + 4\{\,\tfrac{1}{2}, 0\}_c + 6\{0, 0\}$

$$K(\{\mathcal{N} = 4\}) = K(\mathcal{N} = 4\,\text{Maxwell})\,, \qquad K \equiv (E_c, \mathrm{a}, \mathrm{c})$$

$\mathcal{N}$	ϕ	Φ	ψ	Ψ	$T_{\mu\nu}$	V_μ	ψ_μ	$g_{\mu\nu}$	E_c	a	c
1	–	–	–	–	–	1	1	1	$\frac{47}{16}$	3	$\frac{17}{4}$
2	–	–	2	–	1	4	2	1	$\frac{145}{96}$	$\frac{41}{24}$	$\frac{13}{6}$
3	6	–	9	1	3	9	3	1	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$
4	20	2	20	4	6	15	4	1	$-\frac{3}{4}$	-1	-1

Conformal supergravity multiplets

short multiplets with highest spin 2 – 4d conformal supergravity multiplets

$$\mathcal{N} = 3, 4 : \qquad E_c = \tfrac{3}{4}\mathrm{a} \,, \qquad \mathrm{a} = \mathrm{c}$$

Field	$(\Delta;\ j_1, j_2)$	E_c	a	c
$\phi\ (\Box)$	$(3; 0, 0)$	$\frac{1}{240}$	$\frac{1}{360}$	$\frac{1}{120}$
$\Phi\ (\Box^2)$	$(4; 0, 0)$	$-\frac{3}{40}$	$-\frac{7}{90}$	$-\frac{1}{15}$
$\psi\ (\partial)$	$(\frac{5}{2};\ \frac{1}{2}, 0) + (\frac{5}{2};\ 0, \frac{1}{2})$	$\frac{17}{960}$	$\frac{11}{720}$	$\frac{1}{40}$
$\Psi\ (\partial^3)$	$(\frac{7}{2};\ \frac{1}{2}, 0) + (\frac{7}{2};\ 0, \frac{1}{2})$	$-\frac{29}{960}$	$-\frac{3}{80}$	$-\frac{1}{120}$
$T_{\mu\nu}\ (\Box)$	$(3; 1, 0) + (3; 0, 1)$	$\frac{1}{40}$	$-\frac{19}{60}$	$\frac{1}{20}$
$V_\mu\ (\Box)$	$(3;\ \frac{1}{2}, \frac{1}{2})$	$\frac{11}{120}$	$\frac{31}{180}$	$\frac{1}{10}$
$\psi_\mu\ (\partial^3)$	$(\frac{7}{2};\ 1, \frac{1}{2}) + (\frac{7}{2};\ \frac{1}{2}, 1)$	$-\frac{141}{80}$	$-\frac{137}{90}$	$-\frac{149}{60}$
$g_{\mu\nu}\ (\Box^2)$	$(4;\ 1, 1)$	$\frac{553}{120}$	$\frac{87}{20}$	$\frac{199}{30}$

• $\mathcal{N}=4$ CSG + four $\mathcal{N}=4$ Maxwell is anomaly free [Fradkin, AT 81]

$$K(\mathcal{N}=4\,\mathrm{CSG}) + 4\,K(\mathcal{N}=4\,\mathrm{Maxwell}) = 0\,, \qquad\qquad K = (E_c, \mathrm{a}, \mathrm{c})$$

• $\mathcal{N}=4$ CSG multiplet: isomorphic to supercurrent multiplet
of $\mathcal{N}=4$ Maxwell theory and to short massless multiplet
of 5d $\mathcal{N}=8$ sugra with AdS_5 vacuum isometry $PSU(2,2|4)$
• 5d expressions for conf anomaly and Casimir energy for $\mathcal{N}=4$ CSG
are directly related to 1-loop contribution of $\mathcal{N}=8$ 5d supergravity

$$K(\mathcal{N}=4\,\mathrm{CSG}) = -2\,K^+(\mathcal{N}=8\ \mathrm{5d\ SG})$$

this is 1-loop generalization of tree-level relation [Liu, AT 98]
• implies that

$$K^+(\mathcal{N}=8\ \mathrm{5d\ SG}) = 2\,K(\mathcal{N}=4\,\mathrm{Maxwell})$$

• this may be interpreted as expressing the fact that
states of $\mathcal{N}=8$ 5d supergravity are in product of
two $\mathcal{N}=4$ superdoubletons [Gunaydin, Minic, Zagerman 98]

spin (j_L, j_R)	$SU(4)$
(j_1+1, j_2+1)	1
$(j_1+1, j_2+\frac{1}{2}) + (j_1+\frac{1}{2}, j_2+1)$	$4+4^*$
$(j_1+\frac{1}{2}, j_2+\frac{1}{2})$	$1+15$
$(j_1+1, j_2) + (j_1, j_2+1)$	$6+6$
$(j_1+\frac{1}{2}, j_2) + (j_1, j_2+\frac{1}{2})$	$4+4^*+20+20^*$
$(j_1+1, j_2-\frac{1}{2}) + (j_1-\frac{1}{2}, j_2+1)$	$4+4^*$
(j_1, j_2)	$1+15+20'$
$(j_1+\frac{1}{2}, j_2-\frac{1}{2}) + (j_1-\frac{1}{2}, j_2+\frac{1}{2})$	$6+6+10+10^*$
$(j_1+1, j_2-1) + (j_1-1, j_2+1)$	$1+1$

spin (j_L, j_R)
$(j_1, j_2-\frac{1}{2}) + (j_1-\frac{1}{2}, j_2)$
$(j_1+\frac{1}{2}, j_2-1) + (j_1-1, j_2+$
$(j_1-\frac{1}{2}, j_2-\frac{1}{2})$
$(j_1, j_2-1) + (j_1-1, j_2)$
$(j_1-\frac{1}{2}, j_2-1) + (j_1-1, j_2-$
(j_1-1, j_2-1)

General long higher spin massless supermultiplet of $PSU(2,2|4)$

general long massless $\mathcal{N}=4$ superconformal multiplet [Gunaydin et al 98]
has spin range 4: 8 supercharges
conformal representations are massless: $\Delta = 2 + j_1 + j_2$
are of $[j_1, j_2] \oplus [j_2, j_1]$ ($[j_1, j_2]$ in table)
representing massless higher spin fields in AdS_5
or corresponding 4d conformal higher spin fields for all choices of j_1, j_2

$$\mathcal{N} = 4 : \qquad E_c = \mathrm{a} = \mathrm{c} = 0$$

Applications to AdS/CFT

Adjoint AdS_5/CFT_4: 1-loop correction in IIB 10d supergravity on S^5

type IIB superstring on $AdS_5 \times S^5$ and $\mathcal{N}=4$ $SU(N)$ SYM theory

$Z_{\rm SYM}$ on M^4 = $Z_{\rm string}$ on asymptotically AdS_5 with bndry M^4

implies matching of conformal anomalies and Casimir energies

direct comparison possible due to non-renormalization: on SYM side

$$K(\mathcal{N}=4\,{\rm SU(N)\ SYM}) = (N^2-1)\,{\rm k}\,, \qquad\qquad K \equiv (E_c, a, c)$$

${\rm k} = (\frac{3}{16}, \frac{1}{4}, \frac{1}{4})$ for single $\mathcal{N}=4$ Maxwell multiplet

at N^2 order (string tree level – classical type IIB supergravity)

demonstrated in [Henningson, Skenderis 98] (conformal anomalies)

and [Balasubramanian, Kraus 99] (vacuum energy)

string one-loop order: assume contributions of massive string modes vanish

(i) string modes: long $PSU(2,2|4)$ multiplets, should not contribute

(ii) masses depend on 't Hooft coupling ($m^2 \sim \alpha'^{-1} \sim \sqrt{\lambda}$)

contribution would contradict expectred non-renormalization

	$(\Delta;\ j_1, j_2)$	$SU(4)$
$p \geq 2$	$(p;\ 0,0)$	$(0,p,0)$
	$(p+\frac{1}{2};\ \frac{1}{2},0)$	$(0,p-1,1)_c$
	$(p+1;\ 1,0)$	$(0,p-1,0)_c$
	$(p+1;\ 0,0)$	$(0,p-2,2)_c$
	$(p+2;\ 0,0)$	$(0,p-2,0)_c$
	$(p+\frac{3}{2};\ \frac{1}{2},0)$	$(0,p-2,1)_c$
	$(p+1;\ \frac{1}{2},\frac{1}{2})$	$(1,p-2,1)$
	$(p+\frac{3}{2};\ 1,\frac{1}{2})$	$(1,p-2,0)_c$
	$(p+2;\ 1,1)$	$(0,p-2,0)$

	$(\Delta;\ j_1, j_2)$	$SU(4)$
$p \geq 3$	$(p+\frac{3}{2};\ \frac{1}{2},0)$	$(2,p-3,1)_c$
	$(p+\frac{5}{2};\ \frac{1}{2},0)$	$(0,p-3,1)_c$
	$(p+2;\ \frac{1}{2},\frac{1}{2})$	$(1,p-3,1)_c$
	$(p+2;\ 1,0)$	$(2,p-3,0)_c$
	$(p+3;\ 1,0)$	$(0,p-3,0)_c$
	$(p+\frac{5}{2};\ 1,\frac{1}{2})$	$(1,p-3,0)_c$
$p \geq 4$	$(p+2;\ 0,0)$	$(2,p-4,2)$
	$(p+3;\ 0,0)$	$(0,p-4,2)_c$
	$(p+4;\ 0,0)$	$(0,p-4,0)$
	$(p+\frac{5}{2};\ \frac{1}{2},0)$	$(2,p-4,1)_c$
	$(p+\frac{7}{2};\ \frac{1}{2},0)$	$(0,p-4,1)_c$
	$(p+3;\ \frac{1}{2},\frac{1}{2})$	$(1,p-4,1)$

Table 1: Field content of compactification of type IIB supergravity on S^5

$O(N^0)$ term should come from loop of massless string modes:
one-loop correction in 10d type IIB supergravity compactified on S^5
sum of contributions of massless $\mathcal{N}=8$ $5d$ supergravity multiplet
and tower of massive KK multiplets [Kim, Romans, van Nieuwenhuizen 85]
thus should find

1-loop 10d IIB SG on S^5: $\quad E_c^+ = -\frac{3}{16}, \quad \mathrm{a}^+ = -\frac{1}{4}, \quad \mathrm{c}^+ = -\frac{1}{4}$

[contributions of 5d fields with standard ("Dirichlet") b.c.: $K^+ = -\frac{1}{2}K$]

$$K^+(\text{10d IIB SG on S}^5) \;=\; -K(\mathcal{N}=4\ \text{Maxwell})$$

vacuum energy does not vanish in 1-loop type IIB supergravity on S^5
different from $\mathcal{N}>4$ gauged SG in 4d [Allen 83]
and 11d SG on S^7 [Gibbons, Nicolai 84]
but similar to 11d SG on S^4 [Beccaria, AT]
use general expressions for $\mathrm{a}, \mathrm{c}, E_c$ and table of KK states to compute
massless level: states of 5d $\mathcal{N}=8$ SG give ($p=2$)

$$p=2: \qquad E_c = \frac{3}{8}, \qquad \mathrm{a} = \frac{1}{2}, \qquad \mathrm{c} = \frac{1}{2}$$

• same up to -1/2 as of $\mathcal{N}=4$ 4d conformal supergravity multiplet

$p=3$ and $p\geqslant 4$ massive KK multiplets give

$$p\geq 3: \qquad E_c=\frac{3p}{16}\,, \qquad \mathrm{a}=\frac{p}{4}\,, \qquad \mathrm{c}=\frac{p}{4}$$

• $K=(E_c,\mathrm{a},\mathrm{c})$ are thus universally described by ($p=2,3,4,...$)

$$K^{+}(\text{KK level } p \text{ of 10d IIB SG on } \mathrm{S}^5) \;=\; p\,K(\mathcal{N}=4\ \text{Maxwell})$$

• applies also for $p=1$:

$\mathcal{N}=4$ superdoubleton multiplet = Maxwell multiplet

linearity in p: E_c, a and c are 5th order polynomials in $\Delta-2$ (and thus in p)

• non-linearity in p cancels out after multiplying by dimensions of $SO(6)$ reps and summing over the members of each supermultiplet

cf. 5d states at level p appear in product of p $\mathcal{N}=4$ doubletons [Gunaydin]

• how to sum over p: corect prescription

$$\sum_{p=1}^{\infty} p=0\,, \qquad \text{i.e.} \qquad \sum_{p=2}^{\infty} p=-1$$

interpretation: $p = 1$ term – $\mathcal{N} = 4$ Maxwell multiplet = superdoubleton
should not to be included – gauged away
cf. decoupled $U(1)$ D3-brane contribution or $SU(N)$ vs $U(N)$ on SYM side

true if use sharp cutoff $\sum_{p=1}^{P} p = \frac{1}{2}P^2 + \frac{1}{2}P \to 0$
can be justified for E_c by ζ-function regularization directly in 10d
regularization consistent with symmetries of theory

should be applied directly in 10d rather than in 5d:
should be based on spectrum of original 10d operators

Vectorial AdS$_5$/CFT$_4$

no supersymmetry, free CFT at the boundary in any d

$d = 4$ or AdS$_5$: first non-trivial case where mixed-symmetry representations appear in type B and type C theories

type C theory: dual to (complex or real) N 4d Maxwell fields

can be obtained by taking the product of two spin 1 doubletons

complex Maxwell field case: $F^*_{\mu\nu}(x) F_{\kappa\rho}(x') \to F^* \partial ... \partial F$

dimension 4 states $F^*_{..} F_{..}$:

(i) scalar $F^*_{\mu\nu} F^{\mu\nu}$ and pseudoscalar $F^*_{\mu\nu} \widetilde{F}^{\mu\nu}$ in rep $(4; 0, 0)$;

(ii) antisymmetric tensor $F^*_{\mu[\nu} F_{\kappa]\mu}$ – massive selfdual + anti-selfdual rank 2 tensors: $(4; 1, 0)_c = (4; 1, 0) + (4; 0, 1)$

(iii) spin 2 conserved stress tensor $(4; 1, 1)$ and its parity-odd counterpart with one $F_{\mu\nu}$ replaced by $\widetilde{F}_{\mu\nu}$

(iv) conserved current with symmetries of Weyl tensor, i.e. massless state $(4; 2, 0)_c$ described by Young tableu with 2 rows and 2 columns

AdS_5	CFT_4 (singlet sector)
non-minimal type A theory $(2;0,0)+\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N complex scalars : $U(N)$
minimal type A theory $(2;0,0)+\bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})$	N real scalars : $O(N)$
non-minimal type B theory $2\,(3;0,0)+$ $2\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=1}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Dirac fermions : $U(N)$
minimal type B theory $2\,(3;0,0)+$ $\bigoplus_{s=1}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s+1}{2},\frac{s-1}{2})_c$	N Majorana fermions : $O(N)$
non-minimal type C theory $2\,(4;0,0)+(4;1,0)_c$ $2\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N complex Maxwell vectors : $U(N)$
minimal type C theory $2\,(4;0,0)+$ $\bigoplus_{s=2}^{\infty}(2+s;\frac{s}{2},\frac{s}{2})+\bigoplus_{s=2,4,\dots}^{\infty}(2+s;\frac{s+2}{2},\frac{s-2}{2})_c$	N real Maxwell vectors : $O(N)$

Table 2: Vectorial AdS_5/CFT_4 dualities. $(\Delta;j_1,j_2)_c=(\Delta;j_1,j_2)+(\Delta;j_2,j_1)$

sum over spins prescription: sum with fixed cutoff

implied by use of spectral ζ-function

$$\sum_s K(s) \equiv \sum_s e^{-\epsilon\,(s+\frac{1}{2})}\, K(s)\Big|_{\epsilon\to 0,\ \text{finite part}}\ , \qquad K = (E_c, \text{a}, \text{c})$$

$s = j_1 + j_2$ is total spin and summation over all states

non-minimal type A theory:

$$\sum_{s=1}^{\infty} K^{+}(2+s;\ \tfrac{s}{2}, \tfrac{s}{2}) = 0$$

minimal type A theory:

$$\sum_{s=2,4,\dots}^{\infty} K^{+}(2+s;\ \tfrac{s}{2}, \tfrac{s}{2}) = K(3;\ 0,0)$$

i.e. AdS_5 HS theory 1-loop correction is exactly 1-loop contribution
of single real massless 4d scalar: $K(3;\ 0,0) = (\frac{1}{240}, \frac{1}{360}, \frac{1}{120})$
consistent with AdS/CFT duality if minimal HS theory action $N \to N-1$

non-minimal type B theory:

$$2\,K^{+}(3;\,0,0)+2\sum_{s=1}^{\infty}K^{+}(2+s;\,\tfrac{s+1}{2},\tfrac{s-1}{2})=0$$

$2\,K^{+}(3;\,0,0)=-K(3;0,0)$ contribution of two 5d scalars

symmetric representation term vanishes separately

contributions of $(\Delta;j_1,j_2)$ and $(\Delta;j_2,j_1)$ are equal: doubling

minimal type B theory:

$$2\,K^{+}(3;\,0,0)+2\sum_{s=2,4,\dots}^{\infty}K^{+}(2+s;\,\tfrac{s+1}{2},\tfrac{s-1}{2})=K(\tfrac{5}{2};\,\tfrac{1}{2},0)_c$$

r.h.s. is same as contribution of single 4d Majorana fermion

$K(\tfrac{5}{2};\,\tfrac{1}{2},0)_c=2K(\tfrac{5}{2};\,\tfrac{1}{2},0)=(\tfrac{17}{960},\tfrac{11}{720},\tfrac{1}{40})$

non-minimal type C theory:

$$2\,K^+(4;\,0,0) + K^+(4;\,1,0)_c$$
$$+\,2\sum_{s=2}^{\infty} K^+(2+s;\,\tfrac{s}{2},\tfrac{s}{2}) + \sum_{s=2}^{\infty} K^+(2+s;\,\tfrac{s+2}{2},\tfrac{s-2}{2})_c$$
$$=2\,K(3;\,\tfrac{1}{2},\tfrac{1}{2}) = -4\,K^+(3;\,\tfrac{1}{2},\tfrac{1}{2})$$

sum of all AdS_5 1-loop contributions is no longer zero – is twice of $K(3;\,\frac{1}{2},\frac{1}{2}) = (\frac{11}{120},\frac{31}{180},\frac{1}{10})$ – same as of one complex 4d Maxwell field
already in non-minimal type C theory case one needs $N \to N-1$?!

minimal type C theory:

$$2\,K^+(4;\,0,0) + \sum_{s=2}^{\infty} K^+(2+s;\,\tfrac{s}{2},\tfrac{s}{2}) + \sum_{s=2,4,\ldots}^{\infty} K^+(2+s;\,\tfrac{s+2}{2},\tfrac{s-2}{2})_c$$
$$=2\,K(3;\,\tfrac{1}{2},\tfrac{1}{2}) = -4\,K^+(3;\,\tfrac{1}{2},\tfrac{1}{2})$$

here boundary vector field is real –
need shift $N \to N-2$ in the coefficient of the classical HS action

Supersymmetric cases

• supersymmetry not a necessary ingredient in vectorial AdS/CFT duality but may consider also supersymmetric AdS_5/CFT_4 dual pairs (supersymmetric AdS_4/CFT_3 cases [Sezgin, Sundell 03,Leigh, Petkou 03])

• $\mathcal{N}=1$ supersymmetric HS theory in AdS_5 [Alkalaev, Vasiliev 02]
boundary theory – N free spin $(0,\frac{1}{2})$ $\mathcal{N}=1$ supermultiplets
similar susy generalizations of type A, B and C theory examples

• most supersymmetric case of free unitary boundary CFT:
N free $\mathcal{N}=4$ Maxwell supermultiplets

• spectrum of dual AdS_5 HS theory: product of two $\mathcal{N}=4$ superdoubletons [Gunaydin et al 98; Sezgin, Sundell 02]
low-spin $s\leqslant 2$ part same as in type IIB supergravity compactified on S^5

• this HS theory should correspond to "leading Regge trajectory" part of "zero tension" limit of $AdS_5\times S^5$ superstring [Bianchi et al 03]

• particular maximally supersymmetric case of vectorial AdS/CFT duality as a truncation of $g_{\rm YM}=0$ limit of the adjoint AdS/CFT when 5d fields are combined into supermultiplets many cancellations happen

• $K^+ = (E_c^+, \mathrm{a}^+, \mathrm{c}^+)$ for infinite set of HS 5d fields appearing in product of two superdoubletons $\{\mathcal{N}\}$ each representing $\mathcal{N}$-super Maxwell theory

$$K^+(\{\mathcal{N}\} \otimes \{\mathcal{N}\}) = 2\, K(\{\mathcal{N}\}) = 2\, K(\mathcal{N}\text{-Maxwell})$$

r.h.s. is twice the contribution of $\mathcal{N}$-super Maxwell theory or $\mathcal{N}$-superdoubleton
• get direct super-generalization of the relation in type C theory
"anomaly of a product is twice anomaly of a factor":
may be viewed as analog of relation for the characters or partition functions
$\mathcal{Z}(\{\mathcal{N}\} \otimes \{\mathcal{N}\}) = [\mathcal{Z}(\{\mathcal{N}\})]^2$

• admits the following interpretation:
1-loop contribution of states of $\mathcal{N} = 8$ 5d supergravity is already equal to that of two $\mathcal{N} = 4$ Maxwell multiplets; thus all other states appearing in the product $\{\mathcal{N}\} \otimes \{\mathcal{N}\}$ should give zero contribution: they indeed should form massless supermultiplets of $PSU(2, 2|4)$ giving 0 contributions

6d case: tensor multiplet and $AdS_7 \times S^4$ supergravity

[Beccaria, Macorini, AT]

one-loop computation in 11d supergravity on $AdS_7 \times S^4$:
determine 2nd subleading coeff in conf anomaly of
6d (2,0) theory of N coincident M5-branes
dual to M-theory on $AdS_7 \times S^4$ conformal anomaly in 6d

$$\mathcal{A}_6 = \mathrm{a}\, \mathcal{E}_6 + W_6 + D_6 \ , \qquad W_6 = \mathrm{c}_1\, I_1 + \mathrm{c}_2\, I_2 + \mathrm{c}_3\, I_3$$

$I_1 \sim CD^2C + ..., \ I_2, I_3 \sim CCC, \ \ D_6 \sim D^2(...)$
single free 6d tensor multiplet [Bastianelli, Frolov, AT '00]
classical 11d supergravity on S^7 [Henningson, Skenderis 98]:
large N of (2,0) theory

$$\mathcal{A}_6 = \mathrm{a}\, \mathcal{E}_6 + \mathrm{c}\, \mathcal{W}_6 \ , \qquad \mathcal{W}_6 \equiv 96 I_1 + 24 I_2 - 8 I_3 \ ,$$

$$\mathrm{a}_{\rm tens} = \tfrac{7}{4} \ , \quad \mathrm{c}_{\rm tens} = 1 \ , \qquad \mathrm{a}_{(2,0)} = 4\, N^3 + ... \ , \qquad \mathrm{c}_{(2,0)} = 4\, N^3 + ... \ .$$

same Weyl-invariant combination $\mathcal{W}_6$: related to non-renormalization
of ratio of 2- and 3- points of stress tensor [Bastianelli, Frolov, AT 99]

a in 6d related to 4-point stress correlator – gets non-trivial renormalization

as order N term in R-symmetry anomaly [Harvey, Minasian, Moore 98]

order N terms in $\mathrm{a}_{(2,0)}$ and $\mathrm{c}_{(2,0)}$ from R^4 in 11d eff. action [AT '00]

$$\mathrm{a}_{(2,0)} = 4\,N^3 - \tfrac{9}{4}\,N + a_1\ , \qquad\qquad \mathrm{c}_{(2,0)} = 4\,N^3 - 3\,N + c_1$$

by analogy with $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality with anomaly coeff $N^2 - 1$

vanishing for $N = 1$ expect boundary singleton

(single M5-brane tensor multiplet)

should decouple and thus the full 6d anomaly should vanish for $N = 1$:

$$a_1 = -\mathrm{a}_{\rm tens} = -\tfrac{7}{4}\ , \qquad\qquad c_1 = -\mathrm{c}_{\rm tens} = -1$$

$\mathrm{c}_{(2,0)} = 4\,N^3 - 3\,N - 1 = (N-1)(2N+1)^2$ is same as central charge of

A_{N-1} Toda theory at the "symmetric" coupling point

[Beem, Rastelli, van Rees 14]: protected sector – prediction that $c_1 = -1$

(2d chiral algebra)

show that 1-loop 11d supergravity produces expected $a_1 = -\mathrm{a}_{\rm tens}$

$$\mathrm{a}_{(2,0)} = 4\,N^3 - \tfrac{9}{4}\,N - \tfrac{7}{4} = (N-1)\big[(2N+1)^2 + \tfrac{3}{4}\big]$$

1-loop correction in 11d sugra on S^7:

(i) boundary of AdS_7 is S^6 (gives a-anomaly part of $\mathcal{A}_6$)

(ii) $S^1 \times S^5$ (gives Casimir energy $E_c^{1-\text{loop}}$)

result is minus that of single tensor multiplet

$$\mathrm{a}_{1-\text{loop sugra}} = -\mathrm{a}_{\text{tens}}, \qquad E_{c\, 1-\text{loop sugra}} = -E_{c\,\text{tens}} \, .$$

(2,0) tensor multiplet in 6d curved space

5 scalars, 4 MW fermions, self-dual tensor

$$S = \int d^6x \sqrt{g} \Big(-\tfrac{1}{12} H_{ijk}^2 - \tfrac{1}{2} \nabla_i \phi^\alpha \nabla^i \phi^\alpha - \tfrac{1}{10} R \phi^\alpha \phi^\alpha + i \bar{\psi}^I \Gamma_i \nabla^i \psi^I \Big)$$

$$\mathrm{a}_\phi = -\tfrac{1}{72576}, \quad \mathrm{a}_\psi = -\tfrac{191}{1451520}, \quad \mathrm{a}_T = -\tfrac{221}{40320}$$

$$\mathrm{a}_{\text{tens}} = 5\,\mathrm{a}_\phi + 4\,\mathrm{a}_\psi + \mathrm{a}_T = -\tfrac{7}{1152}$$

Single particle thermal partition function

$$\mathcal{Z}(q) = \mathrm{Tr}\, e^{-\beta H} = \sum_n \mathrm{d}_n\, e^{-\beta\, \omega_n} = \sum_n \mathrm{d}_n\, q^{\Delta_n}, \qquad q \equiv e^{-\beta}$$

on $S^1 \times S^5$: [Kutasov, Larsen 00]

$$\mathcal{Z}_\phi = \frac{1}{12} \sum_{n=0}^{\infty} (n+1)(n+2)^2(n+3)\, q^{n+2} = \frac{q^2 - q^4}{(1-q)^6}$$

$$\mathcal{Z}_\psi = \frac{1}{6} \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4)\, q^{n+\frac{5}{2}} = \frac{4\, q^{\frac{5}{2}} - 4\, q^{\frac{7}{2}}}{(1-q)^6}$$

$$\mathcal{Z}_T = \frac{1}{4} \sum_{n=0}^{\infty} (n+1)(n+2)(n+4)(n+5)\, q^{n+3} = \frac{10q^3 - 15q^4 + 6q^5 - q^6}{(1-q)^6}$$

Casimir energy on S^5 [Gibbons, Pope, Perry 06]

$$E_c = \tfrac{1}{2}\,(-1)^F \sum_n \mathrm{d}_n\, \omega_n = \tfrac{1}{2}(-1)^F\, \zeta_E(-1)$$

$$E_{c,\phi} = -\tfrac{31}{60480}, \quad E_{c,\psi} = -\tfrac{367}{96768}, \quad E_{c,T} = -\tfrac{191}{4032}$$

$$E_{c\,\mathrm{tens}} = 5\, E_{c,\phi} + 4\, E_{c,\psi} + E_{c,T} = -\tfrac{25}{384}$$

$\frac{E_{c\,\mathrm{tens}}}{\mathrm{a}_{\mathrm{tens}}} = \frac{75}{7}$ does not agree with [Herzog, Huang 13]:
derivative terms $D_6 \neq 0$ in natural scheme

11d supergravity near $AdS_7 \times S^4$

$SO(2,6) \times SO(5)$: conformal group reps $(\Delta;\, \mathbf{h})$

$\mathbf{h} = (h_1, h_2, h_3)$, $h_1 \geq h_2 \geq |h_3|$ or Dynkin labels $[r_1, r_2, r_3]$

KK spectrum on S^4 [van Nieuwenhuizen 85]

character of typical massive representation [Dolan 05]

$$\mathcal{Z}^+(\Delta;\, \mathbf{h}) \equiv \widehat{\mathcal{Z}}^+(\Delta;\, \mathbf{h}) = \mathrm{d}(\mathbf{h})\, \frac{q^\Delta}{(1-q)^6}$$

$$\begin{aligned}\mathrm{d}(\mathbf{h}) = \tfrac{1}{12}&(1+h_1-h_2)(1+h_2-h_3)\\ &\times(1+h_2+h_3)(2+h_1-h_3)(2+h_1+h_3)(3+h_1+h_2)\end{aligned}$$

singleton representation $\mathbf{h} = (h, h, \pm h)$

$(2,0)$ tensor multiplet as singleton [Gunaydin et al 84]

partition functions on $S^1 \times S^5$ are $h = 0, \frac{1}{2}, 1$ singleton characters

$$\mathcal{Z}^+(2;\, 0,0,0) = \mathcal{Z}_\phi(q), \quad \mathcal{Z}^+(\tfrac{5}{2};\, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \mathcal{Z}_\psi(q), \quad \mathcal{Z}^+(3;\, 1,1,1) = \mathcal{Z}_T(q)$$

	$(\Delta;\ [r_1, r_2, r_3])$	(h_1, h_2, h_3)	$USp(4)$
$p \geq 2$	$(2p; [0,0,0])$	$(0,0,0)$	$(0,p)$
	$(2p+\frac{1}{2}; [1,0,0])$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$(1,p-1)$
	$(2p+1; [2,0,0])$	$(1,1,-1)$	$(0,p-1)$
	$(2p+1; [0,1,0])$	$(1,0,0)$	$(2,p-2)$
	$(2p+\frac{3}{2}; [1,1,0])$	$(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2})$	$(1,p-2)$
	$(2p+2; [0,2,0])$	$(2,0,0)$	$(0,p-2)$
$p \geq 3$	$(2p+\frac{3}{2}; [0,0,1])$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(3,p-3)$
	$(2p+2; [1,0,1])$	$(1,1,0)$	$(2,p-3)$
	$(2p+\frac{5}{2}; [0,1,1])$	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$(1,p-3)$
	$(2p+3; [0,0,2])$	$(1,1,1)$	$(0,p-3)$
$p \geq 4$	$(2p+2; [0,0,0])$	$(0,0,0)$	$(4,p-4)$
	$(2p+\frac{5}{2}; [1,0,0])$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$(3,p-4)$
	$(2p+3; [0,1,0])$	$(1,0,0)$	$(2,p-4)$
	$(2p+\frac{7}{2}; [0,0,1])$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(1,p-4)$
	$(2p+4; [0,0,0])$	$(0,0,0)$	$(0,p-4)$

Casimir energy

$$E_c(\Delta;\,\mathbf{h}) = \frac{(-1)^{2\,(h_1+h_2+h_3)}}{120960}\,\mathrm{d}(\mathbf{h})\,(\Delta-3)$$
$$\times\left[12\,(\Delta-3)^6 - 126\,(\Delta-3)^4 + 336\,(\Delta-3)^2 - 191\right]$$

a–anomaly

$$\mathrm{a}(\Delta;\mathbf{h}) = (-1)^{2(h_1+h_2+h_3)}\,\frac{\mathrm{d}\,(\mathbf{h})}{2\times 96\times 37800}$$
$$\times\left[15(\Delta-3)^7 - 21(\Delta-3)^5\left(h_3^2 + h_1\,(h_1+4) + h_2\,(h_2+2) + 5\right)\right.$$
$$+\,35(\Delta-3)^3\left((h_1+2)^{\,2}\,(h_2+1)^{\,2} + (h_1\,(h_1+4) + h_2\,(h_2+2) + 5)\,h_3^2\right)$$
$$\left.-\,105(\Delta-3)\,(h_1+2)^{\,2}\,(h_2+1)^{\,2}h_3^2\right],$$

for representations saturating unitarity bound need subtractions

One-loop supergravity correction

Casimir energy at level p: summing individual reps contributions

$$E_{c,p} = (6\,p^2 - 6\,p + 1)\,E_{c,\mathrm{tens}}, \qquad p = 2, 3, 4, \ldots.$$

$p = 1$: singleton – true also for $p = 1$: $E_{c,1} = E_{c,\mathrm{tens}}$

a-anomaly:

$$\mathrm{a}_p = (6\,p^2 - 6\,p + 1)\,\mathrm{a}_{\mathrm{tens}}\ , \qquad p = 2, 3, 4, \ldots.$$

again $\mathrm{a}_1 = \mathrm{a}_{\mathrm{tens}}$ and thus $E_{c,p}/E_{c,\mathrm{tens}} = \mathrm{a}_{c,p}/\mathrm{a}_{c,\mathrm{tens}}$

Total contribution: use special regularization

$$\sum_{p=1}^{\infty}(6\,p^2 - 6\,p + 1) = 0, \qquad \text{i.e.} \qquad \sum_{p=2}^{\infty}(6\,p^2 - 6\,p + 1) = -1$$

e.g. use sharp cutoff and drop all power divergences:

$\sum_{p=1}^{\Lambda}(6\,p^2 - 6\,p + 1) = 2\Lambda^3 - \Lambda \to 0$

Proper justification: do not sum KK modes, use ζ-func. reg. directly in 11d

$$\sum_{p=2}^{\infty} E_{c,p} = -E_{c,\mathrm{tens}} \; , \qquad\qquad \sum_{p=2}^{\infty} \mathrm{a}_p = -\mathrm{a}_{\mathrm{tens}}$$

Analytic regularisation for E_c

define energy in 11d with cutoff $\epsilon \to 0$

$$E_c(\Delta, \mathbf{h}) = \tfrac{1}{2}(-1)^{2(h_1+h_2+h_3)}\, \mathrm{d}(\mathbf{h}) \sum_{n=0}^{\infty} \binom{n+5}{5} (\Delta + n)\, e^{-\epsilon(\Delta+n)}$$

$$\sum_{p=1}^{\infty} E_{c,p} = \tfrac{785}{2048\epsilon^3} - \tfrac{5}{16\epsilon^2} + 0 + \mathcal{O}(\epsilon) \quad \to \quad 0$$

equivalent to ζ-regularization in 11d

Conclusions

• quantum tests of vectorial – higher spin AdS/CFT:
general mixed representations in $AdS_{d+1}, \ d = 2, 4, 6$

• supersymmetric examples: cancellations, simple patterns of contrubutions of KK multiplets;
subleading terms in a-anomaly coefficients:
$a_{d=4} = N^2 - 1, \quad a_{d-=6} = 4N^3 - \frac{9}{4}N - \frac{7}{4}, \quad a_{d=2} = 6(N_5 N_1 + 1)$

• applications: to adjoint AdS/CFT in "zero-tension"' limit