Remarks on $d \approx 3$ HS theory

Higher Spin Theories and Holography

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*Based on the work to appear, with G.Gomez, P.Kessel and M.Taronna

Intro

Free fields are boring while HS geometry is not understood well. The second order/cubic action is the approximation where all fields become to interact while all the interaction terms have a plain meaning.

3*d* Vasiliev theory† is a 'toy' model, but still highly nontrivial, and captures many basic features of higher-dimensional cousins. The 3*d* Vasiliev equations are very close to 4*d* and any-*d* ones. Rich AdS/CFT dualities.

†Unless otherwise stated all references are to Vasiliev's works

Intro

Vasiliev HS theories feature quasilocal expressions

$$J_{a(s)} = \sum_k \nabla_a..\nabla_a \nabla_{c(k)} \Phi \nabla_a..\nabla_a \nabla^{c(k)} \Phi$$

which naturally come out of star-products.

At the cubic/second order these are not needed (cubic vertices have a finite number of derivatives) and may not be safe. At the quartic order and higher ∞ of derivative couplings is necessary and the quasi-local expressions can be easily hidden under the carpet. Therefore, classes of functions/redefinitions etc. are easier to answer at the cubic order.

HS fields in $3d$

In $3d$ most of the fields do not propagate, except for maximal depth p.m., $s=0,\frac{1}{2}$, $s=1$ can be dualized.

Instead of Fronsdal fields

$$\phi_{\underline{m}_1\ldots\underline{m}_s}$$

one can use frame-like fields

$$e^{a(s-1)}_{\underline{m}} \qquad \omega^{a(s-1)}_{\underline{m}}=\epsilon^{a}{}_{bc}\omega^{a(s-2)b,c}_{\underline{m}}$$

$so(2,1)\sim sp(2)$ allows to replace them with totally-symmetric spin-tensors

$$e^{\alpha(2s-2)}_{\underline{m}} \qquad \omega^{\alpha(2s-2)}_{\underline{m}}$$

These can be organized as gauge fields of HS algebra

HS algebra

AdS_3 algebra $so(2,2) \sim sp(2) \oplus sp(2)$.

$$[L_{\alpha\alpha}, L_{\beta\beta}] = \epsilon_{\alpha\beta} L_{\alpha\beta} \quad [L_{\alpha\alpha}, P_{\beta\beta}] = \epsilon_{\alpha\beta} P_{\alpha\beta} \quad [P_{\alpha\alpha}, P_{\beta\beta}] = \epsilon_{\alpha\beta} L_{\alpha\beta}$$

One takes harmonic oscillator times Clifford algebra $Cl_{2,0}$

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta} \qquad \phi^2 = 1 \qquad \psi^2 = 1 \qquad \{\phi, \psi\} = 0$$

The AdS_3 algebra are bilinears in $\hat{y}_\alpha$

$$L_{\alpha\beta} = -\frac{i}{4}\{\hat{y}_\alpha, \hat{y}_\beta\} \qquad\qquad P_{\alpha\beta} = \phi L_{\alpha\beta}$$

The HS algebra is the algebra of all functions $f(\hat{y}, \phi, \psi)$

$$\omega(\hat{y}, \phi) = \sum_s \left(\phi e^{\alpha(2s-2)} + \omega^{\alpha(2s-2)}\right) \hat{y}_\alpha ... \hat{y}_\alpha$$

Free fields

Distinguished background solution is given by empty AdS space, which is a flat $so(2,2)$-connection of the HS algebra

$$d\Omega = \Omega^2 \qquad\qquad \Omega = \frac{1}{2}\varpi^{\alpha\alpha}L_{\alpha\alpha} + \frac{1}{2}h^{\alpha\alpha}P_{\alpha\alpha}$$

Free HS fields plus matter are described by

$$d\omega = [\Omega, \omega] \qquad\qquad dC = [\Omega, C]$$

where ω and C are one- and zero-forms valued in the HS algebra

Free fields

ψ gives usual HS and matter fields and a shadow sector

$$\widetilde{\mathrm{D}}\tilde{\omega}\psi = 0 \qquad \mathrm{D}\omega = 0$$

$$\widetilde{\mathrm{D}}C\psi = 0 \qquad \mathrm{D}\tilde{C} = 0$$

$$\mathrm{D} = \nabla - \frac{1}{2}h^{\alpha\alpha}[P_{\alpha\alpha}, \bullet] \qquad \widetilde{\mathrm{D}} = \nabla - \frac{1}{2}h^{\alpha\alpha}\{P_{\alpha\alpha}, \bullet\}$$

- matter fields, scalar, $C(\hat{y} = 0|x)$ and fermion, $C_\alpha(x)\hat{y}^\alpha$
- HS gauge fields, $\omega(\hat{y}, \phi)$
- Killing tensors+constant, $\tilde{C}(\hat{y}, \phi|x)$
- Strange one-forms, $\tilde{\omega}(\hat{y}, \phi|x)$

Free fields

HS gauge fields and matter fields are required by the Gaberdiel-Gopakumar conjecture. Nothing was said about Killing tensors. This is puzzling, especially if all of them interact and they do interact.

In Prokushkin-Vasiliev $3d$ theory $C(0)$ is the parameter of the $3d$ family $hs(\lambda)$ of HS algebras. The HS algebra just defined has $\lambda = 0$.

We do not understand the meaning of the shadow sector and for the second-order computations prefer to disentangle it with the physical one.

Unfolded HS theory

The linear equations can be completed to

$$d\omega = F^{\omega}(\omega, C)$$
$$dC = F^{C}(\omega, C)$$

where the expansion is in matter fields C

$$F^{\omega}(\omega, C) = \mathcal{V}(\omega, \omega) + \mathcal{V}(\omega, \omega, C) + \mathcal{V}(\omega, \omega, C, C) + ...$$
$$F^{C}(\omega, C) = \mathcal{V}(\omega, C) + \mathcal{V}(\omega, C, C) + \mathcal{V}(\omega, C, C, C) + ...$$

and F's are constrained by Frobenius integrability condition $d^2 \equiv 0$, which implies certain gauge symmetry.

Perturbative C-exansion is effectively resummed by Vasiliev equations

Second order

The most general equations at the second order are

$$\begin{aligned}\mathcal{D}\omega_2 &= \omega\star\omega + \mathcal{V}(\Omega,\omega,C) + \mathcal{V}(\Omega,\Omega,C,C)\\ \mathcal{D}C_2 &= [\omega, C]_\star + \mathcal{V}(\Omega, C, C)\end{aligned}$$

where some of the cocycles are explicitly determined by the HS algebra.

On the r.h.s. of $\Box\phi_{\underline{m}_1\dots\underline{m}_s} + ... =$ one finds a generalized stress-tensor

$$\mathcal{V}(\Omega,\Omega,C,C)$$

which should be a usual stress-tensor for $s=2$, but it is not

Cubic action

A canonical way to do quantum computations is to have an action, which we do not. The (at least) cubic action consists of three pieces

$$S = S_{CS} + S_{matter} + S_{int}$$

$$S_{CS} = \frac{k}{4\pi}\int tr\left(\omega\wedge d\omega - \frac{2}{3}\omega\wedge\omega\wedge\omega\right)$$

$$S_{matter} = \frac{1}{2}\int \det|e|\left((\nabla\Phi_i)^2 + m^2\Phi_i^2\right)$$

$$S_{int} = \int tr\left(\omega\wedge\mathcal{J}(\Phi^i,\Phi_i)\right)$$

where $\mathcal{J}$ are canonical s-derivative conserved tensors

$$S_{int} = \sum g_s \int \phi_s\left(\Phi\overleftrightarrow{\nabla}^s\Phi\right)$$

Cubic action

One can compare equations with the action

$$DC_2 = [\omega, C] \qquad \text{vs.} \qquad (\Box - m^2)\Phi = \frac{\delta S_{int}}{\delta \Phi}$$

or equivalently gauge transformations

$$\delta C_2 = [\epsilon, C] \qquad \text{vs.} \qquad \delta\Phi = \frac{\partial \cdot \mathcal{J}}{(\Box - m^2)\Phi}$$

which allows to determine all the couplings. The mass of the scalar is also fixed. Bare cubic approximation leaves these numbers undetermined. Complete cubic action is found!

$3d$ Vasiliev equations

The $3d$ equations are based on $osp(1|2)$

$$\begin{aligned} dW &= W * W \\ dS_\alpha &= [W, S_\alpha]_* \\ dT_{\alpha\beta} &= [W, T_{\alpha\beta}]_* \\ \{S_\alpha, S_\beta\}_* &= T_{\alpha\beta} \\ [T_{\alpha\beta}, S_\gamma]_* &= \epsilon_{\alpha\gamma} S_\beta + \epsilon_{\beta\gamma} S_\alpha \end{aligned}$$

The last two equations are defining relations of $osp(1|2)$. W is a flat connection of a bigger algebra that contains HS algebra.

$$f(y,z) \star g(y,z) = \int du\, dv\, f(y+u, z+u) g(y+v, z-v) e^{(iv^\alpha u_\alpha)}$$

a slightly different (canonical) form is achieved by introducing Hubbard-Stratanovich zero-form B and excluding $T_{\alpha\beta}$

$$\begin{aligned} dW &= W * W \\ dS_\alpha &= [W, S_\alpha]_* \\ dB &= [W, B]_* \\ \{S_\alpha, B\}_* &= 0 \\ S_\alpha * S^\alpha &= 1 + B \end{aligned}$$

There is also a well-known feature of naive perturbation theory not being manifestly Lorentz-covariant. The right Lorentz generators are given by coset

$$\frac{sp(2)_{gl} \ltimes sp(2)_{loc}}{sp(2)_{diag}}$$

Free fields from Vasiliev equations

The $3d$ theory turns out to be even more complicated then the $4d$ one because of

$$\tilde{D}\tilde{\omega}\psi = \frac{1}{8}H^{\alpha\alpha}(y_\alpha + i\partial_\alpha)(y_\alpha + i\partial_\alpha)C(w,\phi)\psi\Big|_{w=0} \qquad D\omega = 0$$

$$\tilde{D}C\psi = 0 \qquad D\tilde{C} = 0$$

that can be eliminated via a change of variable

$$\Delta\tilde{\omega} = \frac{1}{4}\phi h^{\alpha\alpha}\int (t^2-1)(y_\alpha + it^{-1}\partial^y_\alpha)(y_\alpha + it^{-1}\partial^y_\alpha)C(yt,\phi)$$

Note that the source is Φ and $\nabla\Phi$ while the redefinition has $\nabla^\infty\Phi$.

Stress-tensors

Instead of the canonical s-derivative tensors we find a sum of several (many) terms

$$h^{\alpha}{}_{\nu}\wedge h^{\nu\alpha}\int_0^1 dt\, dq\int d\xi d\eta\, P(t,q)(\text{two-ferm}+\text{four-ferm})$$
$$e^{i(ay\xi+by\eta+c\eta\xi)}C(\xi,\phi|x)C(\eta,-\phi|x)$$

which can be rewritten in the index form

$$\sum_{A,B=0}^{A+B\leq 2}\alpha^{n,m,l}_{A,B}H^{\beta(A+B)}{}_{\alpha(2-A-B)}C_{\beta(A)\alpha(n+A-1)\nu(l)}C^{\nu(l)}{}_{\beta(B)\alpha(m+B-1)}$$

where on-shell derivatives of the scalar are parameterized by

$$C^{\alpha(2k)}=\nabla^{\alpha\alpha}...\nabla^{\alpha\alpha}\Phi$$

Locality and redefinitions

The stress-tensor consists of three pieces that are conserved! independently.

It has an unbounded number of derivatives.

A remarkable statement proved by P-V is that canonical s-derivative stress-tensor is exact in AdS

$$H\Phi\nabla^s\Phi = DU$$

where U is quasi-local, i.e. of the same type as the redefinition needed to make stress-tensors into canonical s-derivative stress-tensors.

Redefinitions

Canonical s-derivative currents are quasi-locally exact (P-V)

$$\langle \phi\phi J_s \rangle = \int_{AdS} tr\,(\omega \wedge DK) = -\int_{\partial AdS} tr\,(\omega \wedge K)$$

There is a nontrivial cohomology at degree one (P-V), which is a natural candidate for K and explains a bit why shadow fields may be present.

The physical observables should be independent of redefinitions

$$\langle \phi\phi J_s \rangle = \int_{AdS} tr(\omega \wedge J + DU) = G^{-1}\mathcal{V}(\Omega, \Omega, C, C)$$

Admissible Lagrangian and e.o.m. redefinitions belong to different classes?!

d-dim Vasiliev theory at $d = 3$

Bosonic d-dim theory can be extrapolated to $d = 3$ and the shadow sector can be added via

$$\{y_\alpha, k\} = 0 \qquad \{z_\alpha, k\} = 0 \qquad [y^a_\alpha, k] = 0 \qquad k^2 = 1$$

In contrast to 3d-theory, the shadow sector can be truncated away. In particular, there are now shadow sources that are quadratic in physical fields

In 3d theory we find a nontrivial source

$$D\tilde{C}_2 = \mathcal{V}(\Omega, C, C)$$

i.e. Killing tensors are generated by matter at the second order, but $\delta\lambda$ of $hs(\lambda)$ vanishes. This is puzzling for AdS/CFT

Two Vasiliev theories at $d = 3$

There is a family of $3d$ HS algebras, $hs(\lambda)$. They are all covered by Prokushkin-Vasiliev theory, $C(0) = \lambda$. The mass of the scalar is $m^2 = -1 + \lambda^2$.

In $3d$ we have two theories: the $3d$ one has $\lambda = 0$ and the d-dim. at $d = 3$ has $\lambda = 1$. They are expected to be duals of free fermion and free boson. These values of λ are generic, but the behaviour of the shadow sectors is quite different

Proposal: in d-dim. theory at $d = 3$ one can define a more complicated factorization (which is anyway there) such that the resulting HS algebra is $hs(\lambda) \oplus hs(\lambda)$. This seem to solve the puzzle.