Higher Spin Theory
and Holography
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Mixed-symmetry fields in AdS(5), conformal fields and AdS/CFT

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Plan

1) **Introduction**

2) **Computation of two-point functions**

from AdS via Lagrangian approach

Two gauge conditions:

Modified (Lorentz) de Donder gauge

light-cone gauge

3) **Mixed-symmetry fields in AdS(5)**

General setup of gravity/gauge theory duality

$\mathbf{S_{AdS}(\Phi)}$ **type IIB superstring field action**

$$\Phi = \phi \text{ scalar}$$

$$\phi^{A} \text{ vector}$$

$$\phi^{AB} \text{ tensor}$$

$$\phi^{A_1 \ldots A_s} \text{ arbitrary spin}$$

fields in AdS space

AdS_{d+1}

$$ds^2 = \frac{R^2}{z^2}(dx^a dx^a + dzdz)$$

$\mathbf{x^a}$ boundary flat coordinates

$\mathbf{z}$ radial coordinate

$R = 1$

$$\frac{\delta S_{AdS}}{\delta \Phi} = 0$$

$$\Phi(x,z) \sim z^{d-\Delta} \Phi_{\rm sh}(x)$$

$$\Delta = \frac{d}{2} + \sqrt{m^2 + (s + \frac{d-4}{2})^2}$$

RRM, 2003

Use solution corresponding $\mathbf{\Phi}_{\mathsf{sh}}$

$$S_{AdS}(\mathbf{\Phi}) \equiv S_{\mathsf{eff}}(\mathbf{\Phi}_{\mathsf{sh}})$$

$$\langle \Phi_{\mathsf{cur}}(x_1) \ldots \Phi_{\mathsf{cur}}(x_n) \rangle$$

$$= \frac{\delta^n S_{\mathsf{eff}}}{\delta \Phi_{\mathsf{sh}}(x_1) \ldots \delta \Phi_{\mathsf{sh}}(x_n)}$$

correlation functions from AdS

correlation function from **CFT**

$\phi_{\mathbf{SYM}}$ fields of boundary conformal theory, e.g. SYM

$$S(\phi_{\mathbf{SYM}})$$

$$\Phi_{\mathsf{cur}} = \Phi_{\mathsf{cur}}(\phi_{\mathbf{SYM}})$$

$$\mathbf{V} = \int d^d x \; \mathbf{\Phi}_{\mathsf{sh}}(\mathbf{x})\mathbf{\Phi}_{\mathsf{cur}}(\mathbf{x})$$

$$e^{-\mathbf{S_{cft}}} = \int D\phi_{\mathbf{SYM}} e^{-S(\phi_{\mathbf{SYM}})+\mathbf{V}}$$

AdS/CFT

$$S_{eff}(\Phi_{sh}) \stackrel{?}{=} S_{cft}(\Phi_{sh})$$

Long-term motivation

Computation of $\mathrm{S}_{\mathrm{eff}}(\Phi_{\mathrm{sh}})$ for superstring theory

Find helpful gauge conditions

de Donder like gauge ? (Helpful for computation loop corrections)

Light-cone gauge ? (Green-Schwarz computations in superstring theory)

$S_{\text{eff}}(\Phi_{\text{sh}})$ for low spin fields

via AdS/CFT

	1998 – 1999
scalar field	**GKP, Witten**
massless spin-1	**Freedman et.al.**
massless spin-2	**Liu, Tseytlin**
massive spin-1	**Mueck, Viswanathan**
massive spin-2	**Polishchuk**

Goal

Find

$$S_{\text{eff}}(\Phi_{\text{sh}})$$

for **arbitrary spin** fields

by using AdS/CFT

scalar

$$S = \int d^d x dz \mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}\sqrt{g}(g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2)$$

$$\Phi = z^{\frac{d-1}{2}} \phi$$

scalar

$$\mathcal{L} = \frac{1}{2}|\partial^{\mathbf{a}}\phi|^{\mathbf{2}} + \frac{1}{2}|\mathcal{T}_\nu\phi|^{\mathbf{2}}$$

$$\mathcal{T}_\nu \equiv \partial_z + \frac{\nu}{z}$$

$$\nu = \sqrt{m^2 + \frac{d^2}{4}}$$

scalar

Solution to Dirichlet problem

$$\left(\Box + \partial_{\mathbf{z}}^{\mathbf{2}} - \frac{\nu^2}{z^2}\right)\phi = \mathbf{0}$$

$$\phi(\mathbf{x},\mathbf{z}) \quad \overset{z\to 0}{\longrightarrow} \quad \mathbf{z}^{-\nu+\frac{1}{2}} \ \phi_{\mathsf{sh}}(\mathbf{x})$$

$$\phi(\mathbf{x},\mathbf{z}) \quad \overset{z\to\infty}{\longrightarrow} \quad 0$$

scalar

Solution to Dirichlet problem

$$\phi(\mathbf{x}, \mathbf{z}) = \int d^d y \, \mathrm{G}_\nu(\mathbf{x} - \mathbf{y}, \mathbf{z}) \phi_{\mathsf{sh}}(\mathbf{y})$$

$$\mathrm{G}_\nu(\mathbf{x}, \mathbf{z}) = \frac{z^{\nu + \frac{1}{2}}}{(z^2 + |x|^2)^{\nu + \frac{d}{2}}}$$

scalar

$$S_{\text{eff}} = \int d^d x \, \mathcal{L}_{\text{eff}}|_{z \to 0}$$

$$\mathcal{L}_{\text{eff}} = \phi \mathcal{T}_\nu \phi$$

scalar

Effective action

$$\mathbf{S}_{\text{eff}} = \mathbf{c_0} \int d^d x_1 d^d x_2 \, \mathbf{\Gamma_{12}}$$

$$\mathbf{c_0} = \nu, \qquad \nu = \sqrt{m^2 + \frac{d^2}{4}}$$

$$\mathbf{\Gamma_{12}} = \frac{\phi_{\text{sh}}(x_1)\phi_{\text{sh}}(x_2)}{|x_{12}|^{2\nu+d}}$$

$$|x_{12}|^2 = (x_1 - x_2)^a (x_1 - x_2)^a$$

spin-1

$$\mathcal{L} = -\frac{1}{4} F^{AB} F^{AB}$$

$$F^{AB} = D^A \phi^B - D^B \phi^A$$

spin-1

bulk $\mathbf{so(d,1)} \to$ boundary $\mathbf{so(d-1,1)}$

$$\phi^{\mathbf{A}} \quad = \quad \phi^{\mathbf{a}} \oplus \phi^{\mathbf{z}}$$

$$\phi \equiv \phi^z$$

$$\phi^+ \equiv \phi^0 + \phi^{d-1}$$

Popular (**and important !**) gauge conditions

$\phi = 0$ radial gauge

$\phi^{+} = 0$ light-cone gauge

$\mathbf{D_A \Phi^A = 0}$ **Lorentz gauge**

spin-1

standard Lorentz gauge

$$D^A \phi^A = 0$$

leads to

coupled equations

$$(\Box + \partial_z^2 - \frac{m_1^2}{z^2})\phi^{\mathbf{a}} + \partial^a \phi = 0$$

$$(\Box + \partial_z^2 - \frac{m_0^2}{z^2})\phi + \partial^a \phi^{\mathbf{a}} = 0$$

“Technical” problems with standard

Lorentz and de Donder gauge conditions

1) **Coupled equations**

2) For spin **2, 3, 4, ...**

solutions are expressible

in terms of **Whittaker, Heyn functions**

Little is known about **Heun functions**

asymptotic behavior ???

recurrent relations ???

spin-1

Modified Lorentz gauge

$$\mathbf{D^A}\phi^{\mathbf{A}} + \frac{2}{R}\phi = 0$$

RRM, 1999

Polchinski and Strassler 2001

gives

Decoupled equations

spin-1

Decoupled equations

$$(\Box + \partial_z^2 - \frac{{\nu_{\mathbf{1}}}^2}{z^2})\phi^{\mathbf{a}} = 0$$

$$(\Box + \partial_z^2 - \frac{{\nu_{\mathbf{0}}}^2}{z^2})\phi = 0$$

$$\nu_{\mathbf{1}} = \frac{d-2}{2}, \qquad \nu_{\mathbf{0}} = \frac{d-4}{2}$$

spin-1

Solution to Dirichlet problem

$$\phi^a(x,z) = \int d^d y \mathbf{G}_{\nu_1}(\mathbf{x}-\mathbf{y},\mathbf{z}) \ \phi^a_{\mathsf{sh}}(y)$$

$$\phi(x,z) = \int d^d y \mathbf{G}_{\nu_0}(\mathbf{x}-\mathbf{y},\mathbf{z}) \ \phi_{\mathsf{sh}}(y)$$

$$\mathbf{G}_\nu(\mathbf{x},\mathbf{z}) \equiv \frac{z^{\nu+1/2}}{(z^2+|x|^2)^{\nu+\frac{d}{2}}}$$

spin-1

$$S_{\text{eff}} = \int d^d x \, \mathcal{L}_{\text{eff}}|_{z \to 0}$$

$$\mathcal{L}_{\text{eff}} = \phi^a \mathcal{T}_{\nu_{\mathbf{1}}} \phi^a + \phi \mathcal{T}_{\nu_{\mathbf{0}}} \phi$$

$$\mathcal{T}_\nu = \partial_z + \frac{\nu}{z}$$

spin-1

Effective action

$$S_{\text{eff}} = \int d^d x_1 d^d x_2 \mathbf{\Gamma_{12}}$$

$$\Gamma_{12} = \frac{\phi^a_{\text{sh}}(x_1)\phi^a_{\text{sh}}(x_2)}{|x_{12}|^{2(d-1)}}$$

$$+ \frac{\phi_{\text{sh}}(x_1)\phi_{\text{sh}}(x_2)}{|x_{12}|^{2(d-2)}}$$

Modified Lorentz gauge for bulk AdS fields

$$\mathbf{D^A \phi^A + 2\phi = 0}$$

has left-over gauge symmetry

$$\delta\phi^A = \partial^A \xi$$

$$(\Box + \partial_z^2 - \frac{{\nu_1}^2}{z^2})\xi = 0$$

$$\xi(\mathbf{x}, \mathbf{z}) = \int d^d y G_{\nu_1}(x - y, z)\ \xi^{\mathbf{a}}_{\mathsf{sh}}(\mathbf{y})$$

Modified Lorentz gauge for bulk AdS fields

$$\mathbf{D^A \phi^A + 2\phi = 0}$$

leads to

differential constraint for shadow fields

$$\boldsymbol{\partial^a \phi^a_{\rm sh} + \phi_{\rm sh} = 0}$$

Gauge symmetry of differential constraint

$$\delta\phi^a_{\rm sh} = \partial^a \xi_{\rm sh}$$

$$\delta\phi_{\rm sh} = -\Box\xi_{\rm sh}$$

$$\Gamma_{12} = \phi^a_{\mathsf{sh}}(x_1) \frac{\mathbf{O^{ab}_{12}}}{|x_{12}|^{2(d-1)}} \phi^b_{\mathsf{sh}}(x_2)$$

$$\mathbf{O^{ab}_{12}} = \eta^{\mathbf{ab}} - \mathbf{2} \frac{x^a_{12} x^b_{12}}{|x_{12}|^2}$$

Light-cone frame

$$x^a = x^+, x^-, x^i, \qquad i = 1, \ldots d-2$$

$$x^\pm = x^{d-1} \pm x^0$$

$$\phi^a = \phi^+, \phi^-, \phi^i$$

$$\phi^+_{\mathrm{sh}} = 0 \qquad \text{light-cone gauge}$$

Solution to differential constraint

$$\phi^-_{\mathrm{sh}} = -\frac{\partial^j}{\partial_-}\phi^j_{\mathrm{sh}} - \frac{1}{\partial_-}\phi_{\mathrm{sh}}$$

Light-cone gauge fixed S_{eff}

$$\mathbf{S}_{\text{eff}}^{\text{light}-\text{cone}} = \int d^d x_1 d^d x_2 \; \mathbf{\Gamma}_{\mathbf{12}}^{\text{light}-\text{cone}}$$

$$\Gamma_{12}^{\text{light}-\text{cone}} = \frac{\phi^i_{\text{sh}}(x_1)\phi^i_{\text{sh}}(x_2)}{|x_{12}|^{2(d-1)}}$$

$$+ \frac{\phi_{\text{sh}}(x_1)\phi_{\text{sh}}(x_2)}{|x_{12}|^{2(d-2)}}$$

ϕ^i_{sh}, ϕ_{sh} unconstrained fields

Intermediate Conclusions

1) Use of Modified Lorentz gauge leads to generalized **gauge invariant formulation of CFT**

2) Standard formulation of CFT and light-cone gauge CFT are obtained by using Stueckelberg gauge fixing and light-cone gauge

Spin-2

Einstein equation for h^{AB}

$$D^2 h^{AB} + \ldots = 0$$

Standard de Donder gauge

$$\mathbf{D^B h^{AB}} - \frac{1}{2}\mathbf{D^A h} = 0$$

leads to **coupled EOM**

Spin-2

modified de Donder gauge

RRM, 2008

$$\mathbf{D^B h^{AB}} - \frac{1}{2}\mathbf{D^A h} + \mathbf{2h^{zA}} - \eta^{\mathbf{zA}}\mathbf{h} = 0$$

leads to **decoupled** equations

$$\mathbf{so(d,1)} \quad \Longrightarrow \quad \mathbf{so(d-1,1)}$$

$$h^{AB} \quad = \quad h^{ab} \oplus h^{za} \oplus h^{zz}$$

$$\phi^{\mathbf{ab}} \equiv h^{ab} + \eta^{ab} h^{zz}\,, \quad \phi^{\mathbf{a}} \equiv h^{za}\,, \quad \phi \equiv h^{zz}$$

spin-2: Decoupled equations

$$(\Box + \partial_z^2 - \frac{{\nu_2}^2}{z^2})\phi^{ab} = 0$$

$$(\Box + \partial_z^2 - \frac{{\nu_1}^2}{z^2})\phi^{a} = 0$$

$$(\Box + \partial_z^2 - \frac{{\nu_0}^2}{z^2})\phi = 0$$

$\nu_2 = \frac{d}{2}, \qquad \nu_1 = \frac{d-2}{2}, \qquad \nu_0 = \frac{d-4}{2}$

Spin-2

Solution to Dirichlet problem

$$\phi^{ab}(x,z) = \int d^d y \mathbf{G}_{\nu_2}(\mathbf{x}-\mathbf{y},\mathbf{z}) \ \phi^{ab}_{\mathsf{sh}}(y)$$

$$\phi^{a}(x,z) = \int d^d y \mathbf{G}_{\nu_1}(\mathbf{x}-\mathbf{y},\mathbf{z}) \ \phi^{a}_{\mathsf{sh}}(y)$$

$$\phi(x,z) = \int d^d y \mathbf{G}_{\nu_0}(\mathbf{x}-\mathbf{y},\mathbf{z}) \ \phi_{\mathsf{sh}}(y)$$

$$\mathbf{G}_\nu(\mathbf{x},\mathbf{z}) \equiv \frac{z^{\nu+1/2}}{(z^2+|x|^2)^{\nu+\frac{d}{2}}}$$

Spin-2. Effective action

$$\mathrm{S}_{\text{eff}} = \int d^d x \, \mathcal{L}_{\text{eff}}|_{\mathbf{z}\to\mathbf{0}}$$

$$\mathcal{L}_{\text{eff}} = \phi^{\mathbf{ab}} \mathcal{T}_{\nu_{\mathbf{2}}} \phi^{\mathbf{ab}} + \phi^{\mathbf{a}} \mathcal{T}_{\nu_{\mathbf{1}}} \phi^{\mathbf{a}} + \phi \mathcal{T}_{\nu_{\mathbf{0}}} \phi$$

$$\mathcal{T}_\nu = \partial_z + \frac{\nu}{z}$$

$$\nu_2 = \frac{d}{2}, \qquad \nu_1 = \frac{d-2}{2}, \qquad \nu_0 = \frac{d-4}{2}$$

Spin-2

Effective action

$$\mathrm{S}_{\mathrm{eff}} = \int d^d x_1 d^d x_2 \; \mathbf{\Gamma_{12}}$$

$$\Gamma_{12} = \frac{\phi^{ab}_{\mathrm{sh}}(x_1)\phi^{ab}_{\mathrm{sh}}(x_2)}{|x_{12}|^{2d}}$$

$$+ \frac{\phi^{a}_{\mathrm{sh}}(x_1)\phi^{a}_{\mathrm{sh}}(x_2)}{|x_{12}|^{2(d-1)}}$$

$$+ \frac{\phi_{\mathrm{sh}}(x_1)\phi_{\mathrm{sh}}(x_2)}{|x_{12}|^{2(d-2)}}$$

Modified de Donder gauge for bulk AdS fields

$$\mathbf{D^B h^{AB}} - \frac{1}{2}\mathbf{D^A h} + \mathbf{2h^{zA}} - \eta^{\mathbf{zA}}\mathbf{h} = 0$$

leads to

differential constraints for shadow fields

$$\partial^b \phi_{\rm sh}^{ab} + \partial^a \phi_{\rm sh}^{bb} + \phi_{\rm sh}^a = 0$$

$$\partial^a \phi_{\rm sh}^a + \Box \phi_{\rm sh}^{aa} + \phi_{\rm sh} = 0$$

$\phi_{\rm sh}^{aa}$ can be gauged away

$$\partial^b \phi_{\rm sh}^{ab} + \phi_{\rm sh}^a = 0$$

$$\partial^a \phi_{\rm sh}^a + \phi_{\rm sh} = 0$$

On-shell left-over gauge symmetries of bulk AdS fields lead to gauge symmetries of shadow fields

$$\delta\phi_{\rm sh}^{ab} = \partial^a\xi_{\rm sh}^b + \partial^b\xi_{\rm sh}^a + \eta^{ab}\xi_{\rm sh}$$

$$\delta\phi_{\rm sh}^{a} = \partial^a\xi_{\rm sh} + \Box\xi_{\rm sh}^a$$

$$\delta\phi_{\rm sh} = \Box\xi_{\rm sh}$$

$\phi_{\rm sh}^{aa}$ is Stueckelberg field

$\phi_{\rm sh}^{a}$, $\phi_{\rm sh}$ are not Stueckelberg fields

Arbitrary spin-s AdS field

$$\Phi^{A_1 \ldots A_s}$$

Fronsdal action for free fields

Vasiliev theory of interacting fields

Impose **modified** **de Donder gauge**

$$D^{A}\Phi^{AA_2\ldots A_s} - \frac{1}{2}D^{A_2}\Phi^{AAA_3\ldots A_s}$$

$$+2\Phi^{zA_2\ldots A_s} - \eta^{zA_2}\Phi^{AAA_3\ldots A_s} = 0$$

Decompose

$$so(d,1) \longrightarrow so(d-1,1)$$

$$\Phi^{A_1\ldots A_s} = \Phi^{a_1\ldots a_s}$$

$$\Phi^{a_1\ldots a_{s-1}}$$

$$\ldots\ldots$$

$$\Phi^{a_1a_2}$$

$$\Phi^{a_1}$$

$$\Phi$$

$$\phi^{a_1\dots a_s} = \Phi^{a_1\dots a_s} + \dots$$

$$\phi^{a_1\dots a_{s-1}} = \Phi^{a_1\dots a_{s-1}} + \dots$$

$$\dots\dots$$

$$\phi^{a_1 a_2} = \Phi^{a_1 a_2} + \dots$$

$$\phi^{a} = \Phi^{a_1}$$

$$\phi = \Phi$$

Decoupled equations

$$(\Box + \partial_z^2 - \frac{{\nu_{s'}}^2}{z^2})\phi^{a_1\dots a_{s'}} = 0$$

$$\nu_{s'} = s' + \frac{d-4}{2}$$

$$\phi^{a_1\dots a_{s'}}(x, z) = \int d^d y G_{\nu_{s'}}(x-y, z)\phi_{sh}^{a_1\dots a_{s'}}(y)$$

$$S_{\text{eff}} = \int dx_1^d dx_2^d \mathbf{\Gamma_{12}}$$

$$\mathbf{\Gamma_{12}} = \frac{\phi_{\text{sh}}^{\mathbf{a_1 \dots a_s}} \phi_{\text{sh}}^{\mathbf{a_1 \dots a_s}}}{|x_{12}|^{2(s+d-2)}}$$

$$+ \quad \frac{\phi_{\text{sh}}^{\mathbf{a_1 \dots a_{s-1}}} \phi_{\text{sh}}^{\mathbf{a_1 \dots a_{s-1}}}}{|x_{12}|^{2(s+d-3)}}$$

$$+ \quad \dots\dots\dots\dots$$

$$+ \quad \dots\dots\dots\dots\dots$$

$$+ \quad \frac{\phi_{\text{sh}}^{\mathbf{a}} \phi_{\text{sh}}^{\mathbf{a}}}{|x_{12}|^{2(d-1)}}$$

$$+ \quad \frac{\phi_{\text{sh}} \phi_{\text{sh}}}{|x_{12}|^{2(d-2)}}$$

Use of differential constraints for shadow fields leads to

$$\Gamma_{12} = \phi_{\mathsf{sh}}^{a_1 \ldots a_s}(x_1) \frac{O_{12}^{a_1 b_1} \ldots O_{12}^{a_s b_s}}{|x_{12}|^{2(s+d-2)}} \phi_{\mathsf{sh}}^{b_1 \ldots b_s}(x_2)$$

$$\mathbf{O_{12}^{ab}} = \eta^{\mathbf{ab}} - \mathbf{2}\frac{x_{12}^a x_{12}^b}{|x_{12}|^2}$$

Massless: Normalization factor

$$S_{\text{eff}} = \mathbf{c(s,d)} \int d^d x_1 d^d x_2 \Gamma_{12}$$

RRM, 2009

$$\mathbf{c(s,d)} = \frac{(2s+d-3)(2s+d-4)}{2s!(s+d-3)}$$

$$\mathbf{c(1,d)} = \frac{1}{2}(\mathbf{d-2}) \qquad \textbf{Freedman et.al.}$$

$$\mathbf{c(2,d)} = \frac{d(d+1)}{4(d-1)} \qquad \textbf{Liu, Tseytlin}$$

Generalization to **massive fields** is straightforward

use gauge invariant formulation with **Stueckelberg fields**

$$\mathcal{L} = -\frac{1}{4}F^{AB}F^{AB} - \frac{1}{2}(\partial^A\varphi + m\phi^A)^2$$

$F^{AB} = D^A\phi^B - D^B\phi^A$

$$\delta\phi^A = \partial^A\xi$$

$$\delta\varphi = -m\xi$$

$$\mathbf{D^A\phi^A + m\varphi + 2\phi^z = 0}$$

Massive: Normalization factor

$$S_{\mathrm{eff}} = \mathbf{c(m, s, d)} \int dx_1^d dx_2^d \Gamma_{12}$$

$$\mathbf{c(m, s, d)} = \frac{\kappa(2\kappa + 2\mathbf{s} + \mathbf{d} - 2)}{\mathbf{s}!(2\kappa + \mathbf{d} - 2)}$$

$$\kappa \equiv \sqrt{\mathbf{m}^2 + (\mathbf{s} + \frac{\mathbf{d} - 4}{2})^2}$$

RRM, 2011

Mixed-symmetry fields in AdS(5)

unitary highest weigh representations of so(4,2)

$$so(2) \oplus so(4) \qquad so(4) = su_1(2) \oplus su_2(2)$$

$$\mathbf{E_0, j_1, j_2}$$

$$\mathbf{E_0 > j_1 + j_2 + 1\,, \qquad j_1 j_2 = 0}$$

self-dual massive fields

$$\mathbf{E_0 = j_1 + j_2 + 2\,, \qquad j_1 \neq 0\,, j_2 \neq 0}$$

mixed-symmetry massless fields

$$\mathbf{E_0 > j_1 + j_2 + 2\,, \qquad j_1 \neq 0\,, j_2 \neq 0}$$

mixed-symmetry massive fields

$\mathbf{j_1 \neq j_2}$ mixed-symm

$\mathbf{j_1 = j_2}$ totally-symm

Lagrangian formulations of mixed-symmetry fields

Massless and self-dual massive in AdS_5

light-cone gauge RRM 2002

Massless in AdS_{d+1}

frame-like Alkalaev, Shaynkman, Vasiliev 2005

Massless in AdS_5

frame-like Alkalaev 2005

Massive in AdS_5

light-cone gauge RRM 2004

Massive in AdS_{d+1}

frame like Zinoviev 2009

Light-cone gauge fields in irreps $so(4) = su(2) \oplus su(2)$

$$\phi_{\mathbf{m_1},\mathbf{m_2}}$$

$$m_1 = -j_1, -j_1 + 1, \ldots, j_1 \,, \qquad m_2 = -j_2, -j_2 + 1, \ldots, j_2$$

$$|\phi\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \frac{u_1^{j_1+m_1} v_1^{j_1-m_1} u_2^{j_2+m_2} v_2^{j_2-m_2}}{\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} \phi_{m_1,m_2} |0\rangle$$

Oscillators

$$u_\tau, \ \bar{u}_\tau, \qquad v_\tau, \ \bar{v}_\tau, \qquad \tau = 1, 2$$

$$[\bar{u}_\tau, u_\sigma] = \delta_{\tau\sigma}, \qquad [\bar{v}_\tau, v_\sigma] = \delta_{\tau\sigma}$$

$$\bar{u}_\tau |0\rangle = 0\,, \qquad \bar{v}_\tau |0\rangle = 0\,, \qquad u_\tau^\dagger = \bar{u}_\tau\,, \qquad v_\tau^\dagger = \bar{v}_\tau$$

$$S = \int d^5x \, \mathcal{L} \,, \qquad d^5x \equiv dx^+ dx^- dx^1 dx^2 dz$$

$$\mathcal{L} = \langle \phi | (\Box + \partial_z^2 - \frac{1}{z^2} A) | \phi \rangle$$

$$\Box = 2\partial^+ \partial^- + \partial^i \partial^i \,, \qquad i = 1, 2$$

$$A = \nu^2 - \frac{1}{4}$$

$$\nu = \kappa + \mathbf{S_1} - \mathbf{S_2} \,, \qquad \kappa \equiv \mathbf{E_0} - \mathbf{2}$$

$$S_1 = \frac{1}{2}(N_{u_1} - N_{v_1}) \,, \qquad S_2 = \frac{1}{2}(N_{u_2} - N_{v_2})$$

$$N_{u_\tau} \equiv u_\tau \bar{u}_\tau \,, \qquad N_{v_\tau} \equiv v_\tau \bar{v}_\tau$$

$$\mathcal{L} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \mathcal{L}_{m_1,m_2}$$

$$\mathcal{L}_{\mathbf{m_1,m_2}} = \phi^{\dagger}_{\mathbf{m_1,m_2}} \left(\Box + \partial_{\mathbf{z}}^{\mathbf{2}} - \frac{1}{z^2}((\kappa + \mathbf{m_1} - \mathbf{m_2})^{\mathbf{2}} - \frac{1}{4}) \right) \phi_{\mathbf{m_1,m_2}}$$

Two-point function

$$|\phi(x,z)\rangle = \sigma_\nu \int d^4y\, G_\nu(x-y,z)|\phi_{\mathrm{sh}}(y)\rangle$$

$$G_\nu(x,z) = \frac{c_\nu z^{\nu+\frac{1}{2}}}{(z^2+|x|^2)^{\nu+2}}$$

$$c_\nu \equiv \frac{\Gamma(\nu+2)}{\pi^2\Gamma(\nu)}$$

$$\sigma_\nu \equiv \frac{2^\nu\Gamma(\nu)}{2^\kappa\Gamma(\kappa)}(-)^{S_2}$$

$$-S_{\text{eff}} = 2\kappa c_\kappa \Gamma^{\text{sh}-\text{sh}}$$

$$\Gamma^{\text{sh}-\text{sh}} = \int d^4x_1 d^4x_2\, \mathcal{L}^{\text{sh}-\text{sh}}$$

$$\mathcal{L}^{\text{sh}-\text{sh}} \equiv \langle \phi_{\text{sh}}(x_1)| \frac{f_\nu}{|x_{12}|^{2\nu+4}} |\phi_{\text{sh}}(x_2)\rangle$$

$$f_\nu \equiv \frac{4^\nu \Gamma(\nu+2)\Gamma(\nu+1)}{4^\kappa \Gamma(\kappa+2)\Gamma(\kappa+1)}$$

$$|x_{12}|^2 \equiv x_{12}^a x_{12}^a\,, \qquad x_{12}^a = x_1^a - x_2^a$$

$$\nu = \kappa + \mathbf{S_1} - \mathbf{S_2} \qquad \kappa = \mathbf{E_0} - \mathbf{2}$$

$$\kappa - \kappa_{\text{int}} = -2\varepsilon\,, \qquad \kappa_{\text{int}} - \text{ integer}$$

$$\frac{1}{|x|^{2\nu+4}} \overset{\varepsilon}{\approx}{}^{0} \; \frac{1}{\varepsilon} \Box^{\nu_{\text{int}}} \delta^{(4)}(x)$$

$$\nu_{\text{int}} \equiv \kappa_{\text{int}} + S_1 - S_2$$

$$\Gamma \overset{\varepsilon\sim 0}{\sim} \frac{1}{\varepsilon}\varrho_{\kappa_{\rm int}} \int d^4x\; \mathcal{L}\,,$$

$$\mathcal{L} = \langle\phi|\Box^{\nu_{\rm int}}|\phi\rangle$$

$$\nu_{\rm int} \equiv \kappa_{\rm int} + S_1 - S_2$$

$$\kappa_{\rm int} = \mathbf{j_1 + j_2 + N}, \quad \mathbf{j_1 j_2 = 0}, \quad \mathbf{N = 0, 1, \ldots}$$

self-dual conformal fields

$$\kappa_{\rm int} = \mathbf{j_1 + j_2}, \qquad \mathbf{j_1 j_2 \neq 0}$$

short mix-sym. conformal fields

$$\kappa_{\rm int} = \mathbf{j_1 + j_2 + N}, \quad \mathbf{j_1 j_2 \neq 0}, \quad \mathbf{N = 1, 2, \ldots}$$

long mix-sym. conformal fields

Long conformal field

$$\mathcal{L} = \sum_{\mathbf{m_1=-j_1}}^{\mathbf{j_1}} \sum_{\mathbf{m_2=-j_2}}^{\mathbf{j_2}} \phi^{\dagger}_{\mathbf{m_1,m_2}} \Box^{\mathbf{j_1+j_2+N+m_1-m_2}} \phi_{\mathbf{m_1,m_2}} \qquad N = 1, 2, \dots$$

$$N^{\mathbb{C}}_{DoF} = (2j_1 + 1)(2j_2 + 1)(j_1 + j_2 + N)$$

Short conformal field

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$$

$$\mathcal{L}_1 = \sum_{\mathbf{m=-j_1+1}}^{\mathbf{j_1}} \phi^{\dagger}_{\mathbf{m,j_2}} \Box^{\mathbf{j_1+m}} \phi_{\mathbf{m,j_2}}$$

$$\mathcal{L}_2 = \sum_{\mathbf{m=-j_2+1}}^{\mathbf{j_2}} \phi^{\dagger}_{\mathbf{-j_1,-m}} \Box^{\mathbf{j_2+m}} \phi_{\mathbf{-j_1,-m}}$$

$$N^{\mathbb{C}}_{\mathrm{D.o.F}} = j_1(2j_1 + 1) + j_2(2j_2 + 1)$$