Computing Logarithmic Corrections for Extremal Black Holes

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Introduction

- Black Holes in a quantum theory of gravitation are expected to have entropy.

$$\boxed{S_{BH} = \frac{A}{4G_N}}$$

- This formula is obtained in two approximations:
  - Low energy,
  - Semi-classical.
- A complete theory of quantum gravity will encode corrections to this formula.
- We will focus on a quantum correction of the form

$$\boxed{\delta S \simeq \ln \frac{A}{4G_N}}$$

Why Log Corrections?

- The Area Law is a powerful constraint on quantum gravity.
- A universal result which any microscopic interpretation must reproduce.
- Question: Can we sharpen this constraint?
- In particular, compute quantum corrections to S_{BH} from low–energy physics?
- We will do this for extremal Black Holes.
  - The quantum answer is explicitly known from string theory.
  - More generally, compute quantum entropy by AdS/CFT.
  - Make new predictions!

Quantum Entropy Function

- A new proposal for the quantum entropy of extremal black holes: Quantum Entropy Function. [Sen]
- Exploits the fact that NHG of an extremal black hole is always $AdS_2 \otimes M$.
- Consider the string theory path integral over all configurations that asymptote to the black hole NHG.
- This path integral is divergent because the radial coordinate η of AdS_2 stretches out to infinity.

$$\mathcal{Z}^{\text{string}}_{AdS_2} \simeq e^{C\cdot L+\mathcal{O}\left(L^{-1}\right)}\mathcal{Z}_{\text{finite}}, \quad L \simeq e^{\eta_0}.$$

- Then the proposal is that

$$\boxed{d\left(Q,P\right) = \mathcal{Z}_{\text{finite}}}$$

- $\mathcal{Z}_{\text{finite}}$ is known as the Quantum Entropy Function (QEF).

Introduction

- We evaluate $\mathcal{Z}_{\text{finite}}$ in a saddle–point approximation.
- One saddle–point of the QEF is the near–horizon geometry of the black hole itself.
- Evaluating $\mathcal{Z}_{\text{finite}}$ at this saddle–point produces

$$\boxed{S_{BH} = \ln d\left(Q, P\right) = \ln \mathcal{Z}_{\text{finite}} = \frac{A}{4}}$$

- Which is the Bekenstein–Hawking formula.
- How do we reproduce the Log term?

$$\boxed{\delta S \simeq \ln \frac{A}{4G_N}}$$

- We do a loop expansion about this saddle–point.

Introduction

- Naively, this sounds prohibitive! Infinite number of fields, what order in loop expansion?
- It turns out that the log terms are simple to reproduce!
- The only contribution to the log term comes from
  - massless fields of supergravity,
  - only one–loop fluctuations,
  - Two derivative sector of the action is sufficient.
- The log term is therefore a quantum counterpart of the leading Bekenstein–Hawking answer!
  - It is determined purely from low–energy physics of the black hole
  - It places a strong constraint on any candidate quantum description of black holes.
- Any candidate quantum gravity theory must produce the Bekenstein–Hawking answer, and the log correction.

Introduction

It turns out that the log term computed in this way matches perfectly with the string theory answer.

Theory	Macroscopic	Microscopic	Match
$\mathcal{N} = 4$	0	0	$\surd$
$\mathcal{N} = 8$	-4	-4	$\surd$
$\mathcal{N} = 2$	$\left(2 - \frac{\chi}{24}\right)$	!!	$\surd$(!!)

χ: Euler character of the CY_3 on which 10–d ST is compactified.

A Puzzle

The string theory answer for $d(Q, P)$ takes the form

$$d\left(Q, P\right) \sim e^{\frac{A}{4}} + \sum_{N} e^{\frac{A}{4N}}$$

Question: What is the origin of these terms in the QEF?
Proposal: sum over all spacetimes $\sim$ black hole NHG.
$\mathbb{Z}_N$ orbifolds are natural candidates.

- They are admissible saddle-points of the QEF.
- At the saddle-point $\mathcal{Z}_{finite} = e^{\frac{A}{4N}}$.
- explain exponentially suppressed corrections to $d\left(Q, P\right)$?
- Test: match the log term!

New Saddle Points

Consider the AdS_2 metric in coordinates $\sigma = \cosh\eta$.

$$ds^2 = a^2\left(\frac{d\sigma}{\sigma^2-1} + \left(\sigma^2-1\right)d\theta^2\right), \quad \theta \in [0, 2\pi)$$

Suppose we identify $\theta \mapsto \theta + \frac{2\pi}{N}$.
Also, rescale coordinates on the quotient space, $AdS_2/\mathbb{Z}_N$.

$$\tilde{\sigma} = \frac{\sigma}{N}, \quad \tilde{\theta} = N\theta,$$

Then the metric becomes

$$ds^2 = a^2\left(\frac{d\tilde{\sigma}^2}{\tilde{\sigma}^2 - \frac{1}{N}} + \left(\tilde{\sigma}^2 - \frac{1}{N}\right)d\tilde{\theta}^2\right), \tilde{\theta} \equiv \tilde{\theta} + 2\pi.$$

Hence, this is a new spacetime which is asymptotically AdS_2.
$\Rightarrow$ should be included in the QEF.

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Gaussian Integrals

Consider a Gaussian Integral over a matrix $M_{ij} = \kappa_i \delta_{ij}$.

$$Z = \int \left(\prod_{i=1}^{n} dx_i e^{-\kappa_i x_i^2} \right) = \sqrt{\frac{1}{\prod_{i=1}^{n} \kappa_i}} = \det{}^{-\frac{1}{2}} M.$$

- This is true only if $\kappa_i > 0 \, \forall \, i$.
- What if say $\kappa_n = 0$? i.e. M has a zero mode?

In that case

$$Z = \int \left(\prod_{i=1}^{n-1} dx_i e^{-\kappa_i x_i^2} \right) \int dx_n = \left(\det{}' M \right)^{-\frac{1}{2}} \int dx_n.$$

- We get a determinant over non-zero modes,
- The zero mode contribution has to be analyzed separately.

The One-Loop Determinant

One-loop corrections about a saddle–point are contained in the determinant

$$\mathcal{Z}_{1-\ell} = \det^{-\frac{1}{2}}(D)\,.$$

Further: define the (integrated) heat kernel

$$K(t) = \sum_m d_m e^{-t\kappa_m}.$$

In this case

$$\ln\det D = \int_0^\infty \frac{dt}{t} K(t)$$

Importantly, for us

$$\ln\mathcal{Z} = \frac{1}{2} K(0;t)\ln A + \cdots\,.$$

Only the t^0 term in K(t) contributes to the log term in the QEF saddle–points.

Strategy

Computing $K(t) \Rightarrow$ Solve for Spectrum of D.

- Couple fields to background metric. Then

$$D \simeq \Delta + \frac{c}{a^2}$$

- Turn on background EM fields. These shift eigenvalues, not degeneracies.
- e.g. Modes on S^2 are labelled by a quantum number ℓ

Qty	Flux OFF	Flux ON
degeneracy	$2\ell + 1$	$2\ell + 1$
Eigenvalue	$\ell(\ell+1), \ell(\ell+1)$	$\ell(\ell-1), (\ell+1)(\ell+2)$

- Compute degeneracies, eigenvalues known.

Scalar on $S^2/\mathbb{Z}_N$

Strategy

- Final computation on $AdS_2 \otimes S^2 \Rightarrow$ non–compact.
- Note the analytic continuation from S^2

$$a^2\left(d\chi^2 + \sin^2\chi d\theta^2\right) \mapsto a^2\left(d\eta^2 + \sinh^2\eta d\theta^2\right),$$

  when $a \mapsto ia$, $\chi \mapsto i\eta$.
- Hence compute on $S^2 \otimes S^2$ and analytically continue.
- Also have to impose the $\mathbb{Z}_N$ orbifold.
- Consider the toy example of the scalar on $S^2/\mathbb{Z}_N$ here.
  - $ds^2 = a^2\left(d\psi^2 + \sin^2\psi d\phi^2\right)$
  - $\mathbb{Z}_N$: $\phi \mapsto \phi + \frac{2\pi}{N}$.
  - Two fixed points: $\chi = 0, \pi$.
  - Note: In contrast, AdS_2 has one fixed point, $\eta = 0$.

Scalar on $S^2/\mathbb{Z}_N$

The spectrum of the scalar Laplacian on S^2:

- Eigenvalues: $E_\ell = \ell(\ell+1)$
- Eigenfunctions: $Y_{\ell,m}(\psi,\phi) = P_\ell^m e^{im\phi}, \quad -\ell \le m \le \ell.$

The heat kernel is

$$K(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} 1 \cdot e^{-\frac{t}{a^2}\ell(\ell+1)}$$

The $\mathbb{Z}_N$ orbifold:

- No change in eigenvalues
- Modes restricted to $m = Np, \quad p \in \mathbb{Z}, \quad -\ell \le m \le \ell,$

The degeneracy changes:

$$d_\ell = \sum_{m=-\ell}^{\ell} \delta_{m,Np}.$$

Scalar on $S^2/\mathbb{Z}_N$

We will use the following representation for δ

$$\delta_{m,Np} = \frac{1}{N}\sum_{s=0}^{N-1} e^{i\frac{2\pi ms}{N}}$$

Then the heat kernel on $S^2/\mathbb{Z}_N$ is

$$K(t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{1}{N}\sum_{s=0}^{N-1} e^{i\frac{2\pi s}{N}m}\right)\cdot\, e^{-\frac{t}{a^2}\ell(\ell+1)}$$

Doing the sum over m

$$K(t) = \frac{1}{N}\sum_{\ell=0}^{\infty}\sum_{s=0}^{N-1}\frac{\sin\frac{(2\ell+1)\pi s}{N}}{\sin\frac{\pi s}{N}} e^{-\frac{t}{a^2}\ell(\ell+1)}$$

Scalar on $S^2/\mathbb{Z}_N$

Degeneracy of E_ℓ on $S^2/\mathbb{Z}_N$:

$$d_\ell = \frac{2\ell+1}{N} + \frac{1}{N}\sum_{s=1}^{N-1}\chi_\ell\left(\frac{\pi s}{N}\right)$$

χ_ℓ is the Weyl character of SU(2).

The heat kernel on $S^2/\mathbb{Z}_N$ is given by

$$K_{S^2/\mathbb{Z}_N}(t) = \frac{1}{N}K_{S^2} + \frac{N^2-1}{6N} + \mathcal{O}(t)\,.$$

Log Term:

$$\boxed{K_{S^2/\mathbb{Z}_N}(0;t) = \frac{1}{3N} + \frac{N^2-1}{6N}}$$

The Analytic Continuation to AdS_2

The Heat Kernel on $\mathrm{S}^2/\mathbb{Z}_N$ has the form

$$K_{\mathrm{S}^2/\mathbb{Z}_N}(t) = \frac{1}{N} K_{\mathrm{S}^2} + \text{conical terms}.$$

To analytically continue to AdS_2,

- $K_{\mathrm{S}^2} \mapsto K_{\mathrm{AdS}_2}$,
- $a \mapsto ia$ in conical terms
- multiply conical terms by half.

This because S^2 has two fixed points, AdS_2 has one. We then find

$$\boxed{K_{\mathrm{AdS}_2/\mathbb{Z}_N}(t) = \frac{1}{N} K_{\mathrm{AdS}_2} + \frac{1}{2}\frac{N^2-1}{6N} + \mathcal{O}(t)}.$$

This is how we compute on the NHG as well.

Zero Mode Contribution

- In principle, the zero mode integral can also contribute.
- If ϕ has n^0_ϕ zero modes, then

$$\mathcal{Z}^{zero}_{str.} = A^{\frac{\beta_\phi}{2} n^0_\phi} \mathcal{Z}_0 .$$

- Suppose Ψ_i is the set of orthonormal zero modes of D.

$$n^0_\phi = \sum_i \langle \Psi_i | \Psi_i \rangle = \sum_i \int_{\mathrm{AdS}_2} \Psi^*_i \Psi_i$$

- To compute on the $\mathbb{Z}_N$ orbifold, project onto orbifold invariant modes.

Counting Zero Modes

Zero modes of Hodge Operator on vector field on AdS_2

$$\mathcal{A}_\mu = \partial_\mu \Phi; \quad \Phi = \left(\frac{\sinh\eta}{1+\cosh\eta}\right)^{|m|} e^{im\theta}$$

Then

$$n_0 = \sum_m \langle \mathcal{A} | \mathcal{A} \rangle \simeq \frac{1}{2N} e^{\eta_0} - 1 + \mathcal{O}(\eta_0).$$

The number of zero modes is the $\mathcal{O}(1)$ term

$$\boxed{n_0 = -1}$$

Final Answers

Theory	Macroscopic	Microscopic	Match
$\mathcal{N} = 4$	0	0	$\surd$
$\mathcal{N} = 8$	-4	-4	$\surd$
$\mathcal{N} = 2$	$\left(2 - \frac{N\chi}{24}\right)$	??	??

The $\mathcal{N} = 2$ answer is interesting and puzzling.

- In $\mathcal{Z}_{\mathbb{Z}_N} \sim \frac{A}{N} + N \ln A$. If $N \simeq \sqrt{A_H}$ then the 1-loop correction is bigger than the classical answer!
- Also, the N dependence does not appear for $\mathcal{N} = 4$ and $\mathcal{N} = 8$. Reproduce from the microscopic side?

The Twisted QEF

The QEF computes the total number of black hole microstates.
Can we extract more refined information? In particular:

- If the theory admits discrete symmetries, compute indices weighted with these symmetries?
- Can we define quantities that behave like indices in theories with less supersymmetry?

Twisted Indices in String Theory do this job.
$\Rightarrow$ if the theory has a discrete symmetry $g \equiv \mathbb{Z}_N$
Compute the Black Hole entropy index, with an insertion of g.

$$\mathrm{Tr}\left[g\,(-1)^h\,(2h)^{2n}\right]$$

$\Rightarrow$ Twisted Index
Question: QEF Interpretation?

The Twisted QEF

- Proposal: QEF, but $\mathbb{Z}_N$ twisted boundary conditions.
- The Black Hole NHG is not an admissible saddle–point.
- The $(NHG)/\mathbb{Z}_N$ is an admissible saddle–point. Indeed

$$\mathcal{Z}_{twisted} \simeq e^{\frac{A}{4N}},$$

which is in accordance with microscopic results.
- Can we match the log term? $\Rightarrow$ K(t) with twisted b.c.
- Yes! For g preserving $\mathcal{N} = 4$ supersymmetry,

Theory	Macroscopic	Microscopic	Match
$\mathcal{N} = 4$	0	0	$\surd$
$\mathcal{N} = 8$	0	0	$\surd$

Conclusions

- The QEF computes the full quantum entropy of extremal black holes.
- We tested this against the string answer for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ black holes.
- The answer for $\mathcal{N} = 2$ black holes has curious properties. It would be interesting to better understand them.
- We also provided evidence that twisted indices can be computed by a QEF approach.
- Again, the matching persists to the quantum level.
- What about indices preserving $\mathcal{N} = 2$ supersymmetry?

Thank You