Uniformizing higher-spin equations

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Based on:

Alkalaev, M.G., Skvortsov, : arXiv:1409.6507, J. Phys. A, to appear

MG 2012, 2006, Barnich, M.G., 2006

Bekaert, MG 2013

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Vasiliev's interacting higher spin (HS) theory

Vasiliev, 1989,. . . , 2003

- Given in the form of equations of motion:

$$
\begin{aligned}
dW + W \star W &= 0 \\
dS_\alpha + [W, S_\alpha]_\star &= 0 \\
dB + W \star B - B \star \widetilde{W} &= 0 \\
S_\nu \star S^\nu - 2(1 + B \star \kappa) &= 0 \\
\{S_\alpha, B \star \kappa\}_\star &= 0
\end{aligned}
$$

All fields take values in a certain star product algebra. κ – Klein element, $\widetilde{(\cdot)}$ – twist automorphism (originates from $so(d-1,1) \subset so(d-1,2)$ embedding)

- Equations: free differential algebra + algebraic constraints. Unfolded form.

- Linearized over AdS vacuum: reproduces massless fields of all spins.

The system is lacking underlying algebraic/geometrical principle

Why HS?

- String (Field) Theory is HS (though massive). Unbroken phase.
- Tractable example of AdS/CFT. Vasiliev system is dual to free conformal QFT (essentially scalar) though other fixed points of $O(N)$ models are also accessible.
- Improvement of gravity theory?

Family of gauge theories

- Lie (super)algebra $\mathfrak{g}$
- Associative superalgebra $\mathcal{A}$
- Space-time with coordinates x^μ. De Rham $d = dx^\mu \frac{\partial}{\partial x^\mu}$

If e_a basis in $\mathfrak{g}$ with $[e_a, e_b] = -(-1)^{|a||b|}[e_b, e_a]$, $|a|$ - Grassmann degree of e_a:

Fields: $\mathcal{A}$-valued 1 form $W(x)$ and 0-forms $T_a(x)$, EOM's: Equations

$$dW + W \star W = 0\,,$$

$$dT_a + [W, T_a]_\star = 0\,,$$

$$[T_a, T_b]^\pm_\star - \mathcal{C}^c_{ab} T_c = 0\,.$$

M.G. 2012, Alkalaev,M.G., Skvortsov 2014

$[T_a, T_b]^\pm_\star = T_a \star T_b - (-1)^{|a||b|} T_b \star T_a$.

If $\{E_A\} = \mathcal{A}$, $|T^A_a| = |e_a| + |E_A|$, $|W^A_\mu| = |E_A|$. Gauge symmetries:

$$\delta W = d\xi + [W, \xi]_\star\,, \qquad \delta T_a = [T_a, \xi]_\star\,,$$

$\xi = \xi(x)$ – $\mathcal{A}$-valued gauge parameter.

Vasiliev equations in various dimensions can be cast into this form

In the HS context:

- $\mathfrak{g}$ is typically $osp(1|2)$ or its extension. Models based on $u(1), sp(2)$ are also known *Vasiliev (2005), M.G. (2006)*

- $\mathcal{A}$ is a $\star$-product algebra extending a relevant HS algebra.

General properties

The system is background independent: no background fields are involved in its definition.

Physical interpretation requires choosing vacuum solution W^0, T_a^0 to expand around and crucially depends on the choice.

Global symmetries: Gauge transformations preserving vacuum

$$d\xi + [W^0, \xi]_\star = 0\,, \qquad [T_a^0, \xi]_\star = 0$$

so that global symmetries form subalgebra $\mathfrak{B} \subset \mathcal{A}$ of $\mathfrak{g}$-invariants (usually the relevant HS algebra).

Algebraic structure

The nontrivial content is in

$$[T_a, T_b]_\star^{\pm} = \mathcal{C}_{ab}^c T_c$$

Indeed, the remaining equations only determine space-time dependence through introduction of the flat connection W.

It is convenient to forget about x-dependence of fields and gauge parameters so that x-independent gauge parameter determine a gauge symmetry

$$\delta_\xi T_a = [T_a, \xi]_\star\,, \qquad \xi \in \mathcal{A}$$

Above x-independent system actually encodes everything.

The equations, its gauge and global symmetries have a simple algebraic interpretation: $\tau : \mathfrak{g} \to \mathcal{A}$ a map from $\mathfrak{g}$ to $\mathcal{A}$. Then $T_a = \tau(e_a)$ (component form) and

$$[\tau(f), \tau(g)]^{\pm}_{\star} = \tau([f,g]), \quad f, g \in \mathfrak{g} \quad \Leftrightarrow \quad [T_a, T_b]^{\pm}_{\star} = \mathcal{C}^{c}_{ab} T_c$$

It is natural to consider equivalent the two maps related by inner automorphism:

$$\tau \sim U_{\xi} \circ \tau \qquad U_{\epsilon}(F) = \exp_{\star}(\epsilon) \star F \star \exp_{\star}(-\epsilon), \quad F \in \mathcal{A}$$

Inifinitesimally

$$\tau(f) \sim \tau(f) + [\tau(f), \xi]_{\star} \quad \Leftrightarrow \quad T_a \sim T_a + [T_a, \xi]_{\star}$$

The inequivalent configurations of the gauge theory we are dealing with are inequivalent homomorphisms $\mathfrak{g} \to \mathcal{A}$.

Linearized system

Given vacuum solution W^0, T_a^0

$$W = W^0 + w, \qquad T_a = T_a^0 + t_a$$

Linearized system

$$D_0 w = 0\,, \qquad \delta w = D_0 \xi\,,$$

$$D_0 t_a + [w, T_a^0]_\star = 0\,,$$

$$[t_a, T_b^0]_\star^\pm + [T_a^0, t_b]_\star^\pm - \mathcal{C}_{ab}^c t_c = 0\,, \qquad \delta t_a = [T_a^0, \xi]_\star\,,$$

where $D_0 = d + [W^0, \cdot]_\star^\pm$ – background covariant derivative, $D_0^2 = 0$.

Introducing:

$$\Delta = c^a [T_a, \cdot]_\star - \frac{1}{2} c^a c^b \mathcal{C}_{ab}^c\,, \qquad \psi = c^a t_a$$

$$\Delta\psi = 0\,, \qquad \delta_\xi \psi = \Delta\xi \qquad \mathfrak{g}\text{-cohomology}$$

Furthermore,

$$\Psi = w + c^a t_a\,, \qquad \Omega = D_0 + \Delta$$

The linearized system:

$$\Omega\Psi = 0\,, \qquad \delta_\xi \Psi = \Omega\xi \qquad \text{BRST first-quantized form}$$

Systems of this type are known (parent form). *Barnich, M.G., et all (2004,2006)*

Let r_a parametrize Δ-cohomology.

$$D_0 w = 0\,, \qquad D_0 r_a = ad_a\, w\,, \qquad ad_a = [T_a^0, \cdot]_\star$$

Can be solved for w modulo $\mathbb{H}^0(\mathfrak{g}, \mathcal{A})$ (i.e. HS algebra): $w = \omega + ad_a^{-1} D_0 r_a$ and $ad_a\, \omega = 0$. Note that ω can be seen as taking values in $\mathbb{H}^0(\mathfrak{g}, \mathcal{A})$. Then

$$D_0\omega = -D_0\, ad_a^{-1} D_0 r_a$$

Expresses the linearized curvature of ω (HS connection) in terms of r_a.

Familiar example (spin 2):

$$D_0 e^a = 0\,, \qquad D_0\omega = e_0^c \wedge e_0^d C^{ab}_{cd}$$

Cohomological interpretation

Vacuum symmetries

$$D_0\xi = 0\,, \qquad [T_a^0, \xi]_\star = 0$$

ξ is parametrized by $\mathcal{B} = \mathbb{H}^0(\mathfrak{g}, \mathcal{A})$ (centralizer of T_a^0 in $\mathcal{A}$). Curvatures are parametrize by $\mathbb{H}^1(\mathfrak{g}, \mathcal{A})$.

If $\mathbb{H}^1(\mathfrak{g}, \mathcal{A}) = 0$ – no degrees of freedom becuase T_a can be set to its vacuum value T_a^0 perturbatively. E.g. if $\mathfrak{g}$-semi-simple Lie algebra and $\mathcal{A}$ decomposes into finite-dimensional modules then *Whitehead* lemma says $\mathbb{H}^1(\mathfrak{g}, \mathcal{A}) = 0$.

In the Vasiliev theories nontriviality of $\mathbb{H}^1(\mathfrak{g}, \mathcal{A})$ relies on specific choice of functional class of symbols and star product in $\mathcal{A}$.

Example: minimal 3d theory

Take: $\mathfrak{g} = osp(1|2)$ and $\mathcal{A}$ to be the relevant HS algebra. More precisely $\mathcal{A} = \mathcal{A}_0 \otimes Cliff(2)$.

$\mathcal{A}_0$ is a $\star$-product algebra of functions in y_α, z_α with *Vasiliev, 1990*

$$(f \star g)(y,z) = f(y,z) \exp\left(\frac{\overleftarrow{\partial}}{\partial y^\alpha} + \frac{\overleftarrow{\partial}}{\partial z^\alpha}\right) \epsilon^{\alpha\beta} \left(\frac{\overrightarrow{\partial}}{\partial y^\beta} - \frac{\overrightarrow{\partial}}{\partial z^\beta}\right) g(y,z) \, .$$

Nontrivial part of the system:

$$\frac{1}{4}\{S_\alpha, S_\beta\}_\star = T_{\alpha\beta}\,, \quad [T_{\alpha\beta}, S_\gamma]_\star = \epsilon_{\alpha\gamma} S_\beta + \ldots\,, \quad [T_{\alpha\beta}, T_{\gamma\delta}]_\star = \epsilon_{\beta\gamma} T_{\alpha\delta} + \ldots$$

can be rewritten as

$$\frac{1}{2} S_\nu \star S^\nu = 1 + B \star \kappa\,, \qquad \{S_\alpha, B \star \kappa\}_\star = 0$$

where

$$B \star \kappa \equiv \frac{1}{2}(S_1 \star S_2 - S_1 \star S_2) - 1$$

d-dimensional system

- Algebra $\mathcal{A} = \mathcal{A}_0 \otimes A_d$, where A_d is the Weyl $\star$-product algebra generated by Y^a_α, $a = 0, \ldots, d-1$, $\alpha = 1, 2$

$$(f \star g)(Y) = f(Y) \exp \left(\frac{\overleftarrow{\partial}}{\partial Y^a_\alpha} \epsilon_{\alpha\beta} \eta^{ab} \frac{\overrightarrow{\partial}}{\partial Y^b_\beta} \right) g(Y) \, .$$

One can introduce Y^A_α, $A = 0, \ldots, d$ through

$$Y^d_\alpha = y_\alpha \, , \qquad \text{More invariantly:} \quad y_\alpha = Y^A_\alpha V_A$$

where $V^A = \delta^A_d$ the compensator.

Note: $\star$-product has a nontrivial symmetric part involving z_α and $y_\alpha = Y^A_\alpha V_A$.

- Lie superalgebra $\mathfrak{g} = sp(2) \ni\!\!\!\!+ osp(1|2)$. If $F_{\alpha\beta} = F_{\beta\alpha}$ are associated to $sp(2)$ and $S_\alpha, T_{\alpha\beta}$ to $osp(1,2)$:

$$[F_{\alpha\beta}, F_{\gamma\delta}]_\star = \epsilon_{\beta\gamma} F_{\alpha\delta} + \ldots\,,$$

$$[F_{\alpha\beta}, T_{\gamma\delta}]_\star = \epsilon_{\beta\gamma} T_{\alpha\delta} + \ldots, \qquad [F_{\alpha\beta}, S_\gamma]_\star = \epsilon_{\alpha\gamma} S_\beta + \ldots$$

$$\frac{1}{4}\{S_\alpha, S_\beta\}_\star = T_{\alpha\beta}\,, \quad [T_{\alpha\beta}, S_\gamma]_\star = \epsilon_{\alpha\gamma} S_\beta + \ldots\,, \quad [T_{\alpha\beta}, T_{\gamma\delta}]_\star = \epsilon_{\beta\gamma} T_{\alpha\delta} + \ldots$$

- vacuum solution

$$S^0_\alpha = z_\alpha\,, \qquad T^0_{\alpha\beta} = z_\alpha z_\beta\,, \qquad F^0_{\alpha\beta} = Y^A_\alpha Y_{A\beta} + z_\alpha z_\beta\,, \qquad W = W^0$$

$W^0 = \omega_{AB}(x) Y^A_\alpha Y^{\alpha B}$ – flat $o(d-1,2)$-connection describing AdS_d

If $F_{\alpha\beta}$ are set to vacuum value $F^0_{\alpha\beta}$ by hands, the equations imlpy:

$$[W, F^0_{\alpha\beta}]_\star = 0\,, \qquad [F^0_{\alpha\beta}, S_\gamma]_\star = \epsilon_{\beta\gamma} S_\alpha + \epsilon_{\alpha\gamma} S_\beta$$

i.e. W – off-shell HS algebra.

Eliminating $T_{\alpha\beta}$ and introducing $B \star \kappa = 1 - \frac{1}{2} S_\nu \star S^\nu$ one gets

$$\begin{aligned} dW + W \star W &= 0 \\ dS_\alpha + [W, S_\alpha]_\star &= 0 \\ dB + W \star B - B \star \widetilde{W} &= 0 \\ S_\nu \star S^\nu - 2(1 + B \star \kappa) &= 0 \\ \{S_\alpha, B \star \kappa\}_\star &= 0 \end{aligned}$$

This is the off-shell system from *Vasiliev 2003*

Extended system

Now keep $F_{\alpha\beta}$ associated to $sp(2)$-factor dynamical

Convenient to use generic basis in $sp(2)$ so that $[e_i, e_j] = \mathcal{C}_{ij}^k e_k$. Linearization of

$$[F_i, F_j]_\star = \mathcal{C}_{ij}^k F_k$$

around $F_i = F_i^0$ gives ($F_i = F_i^0 + f_i$)

$$[F_i^0, f_j]_\star + [f_i, F_j^0]_\star = \mathcal{C}_{ij}^k f_k \qquad \delta f_i = [F_i^0, \xi]_\star \,.$$

$sp(2)$-cohomology problem.

Observation: $[F_{\alpha\beta}^0, \cdot]_\star$ is of vanishing homogeneity in Y, z. Whitehead lemma applies and one can set $F_{\alpha\beta} = F_{\alpha\beta}^0$ using EOM's and gauge transformations

The nontriviality of $[T_a, T_b]_\star = \mathcal{C}_{ab}^c T_c$ based on semisimple algebra can only enter through essentially inifinite-dimensional $\mathcal{A}$. E.g. always trivial in polynomials.

Alternative interpretation of the extended system

If $\mathcal{A}$ is a $\star$-product algebra there are 3 important choices to be made:

1. symmetric part of star-product
2. functional class of symbols
3. choice of vacuum

All 3 are intertwined as e.g. depending on the choice of functional class different vacuum solutions can be (in)equivalent. The same applies to star-products.

An explicit and fruitful illustration: extended system based on $osp(1|2) \ni sp(2)$ but usual Weyl star product (trivial symmetric part).

Consider linearization of $\{S_\alpha, B \star \kappa\} = 0$ around $S_\alpha = z_\alpha$. Denoting $S_\alpha = z_\alpha + A_\alpha$ one gets

$$\{z_\alpha, \epsilon^{\beta\gamma} \frac{\partial}{\partial z^\beta} A_\gamma\} \quad \rightarrow \quad \epsilon^{\beta\gamma} \frac{\partial}{\partial z^\beta} A_\gamma = 0$$

(the triviality of symmetric part of $\star$-product is crucial here)

Using linearized gauge symmetry $\delta_\xi A_\alpha = \frac{\partial}{\partial z_\alpha}\xi$, A_α can be set to zero.

With this choice of $\star$-product Vasiliev system is trivial (as we have seen $F_{\alpha\beta}$ can also be eliminated.)

However, we are still able to play with the vacuum!

Off-shell parent system

Upon elimination of A_α fields $T_{\alpha\beta}$ are also set to vacuum values and only $F_{\alpha\beta}$ remains. The resulting system is determined by $\mathfrak{g} = sp(2)$ and $\mathcal{A}$ beeing Weyl algebra generated by Y^A_α.

$$dW + W \star W = 0\,, \quad dF_i + [A, F_i]_\star = 0\,, \qquad [F_i, F_j]_\star - C^k_{ij} F_k = 0$$

Gauge symmetries

$$\delta_\xi F_i = [F_i, \xi]_\star\,, \qquad \delta_\xi A = d\xi + [A, \xi]_\star$$

With the standard choice of vacuum the system is empty.

Can be made nonempty by chosing as $\mathcal{A}$ polynomials in $Y^A_2 \equiv P^A$ with coefficients in formal series in $Y^A_1 \equiv Y^A$ and taking inhomogeneous vacuum: *M.G. 2012*

$$F^0_{11} = \frac{1}{2}(Y+V)\cdot(Y+V)\,, \qquad F^0_{12} = (Y+V)\cdot P\,, \qquad F^0_{22} = \frac{1}{2}P\cdot P$$
$$W^0 = \omega_{AB}(Y^A + V^A)P^B$$

where $V^A = const^A$, are components of the compensator satisfying $V^A V_A = -1$.

Linearized equations read as

$$dw + [W^0, w]_\star = 0\,, \qquad [F^0_{\alpha\beta}, f_{\gamma\delta}]_\star + [f_{\alpha\beta}, F^0_{\gamma\delta}]_\star = \epsilon_{\alpha\gamma} f_{\beta\delta} + \epsilon_{\alpha\delta} f_{\beta\gamma}$$

On-shell version of this system is known as parent form of massless fields on AdS *Barnich, M.G. 2006*.

System is non-topological. Whithead Lemma does not apply becuase of inifinite-dimensional modules involved!

Nontrivial part of linearized equations

$$[F^0_{\alpha\beta}, f_{\gamma\delta}]_\star + [f_{\alpha\beta}, F^0_{\gamma\delta}]_\star = \epsilon_{\alpha\gamma} f_{\beta\delta} + \epsilon_{\alpha\delta} f_{\beta\gamma}$$

How to see HS fields?

In fact: EOM's and symmetry $\delta_\xi f_{\alpha\beta} = [F^0_{\alpha\beta}, \cdot]_\star$ allow to set $f_{11} = f_{12} = 0$.

One then stays with just:

$$(X \cdot \frac{\partial}{\partial X} - P \cdot \frac{\partial}{\partial P} + 2)\Phi = 0\,, \qquad X \cdot \frac{\partial}{\partial P}\Phi = 0\,, \qquad \Phi \sim \Phi + P \cdot \frac{\partial}{\partial X}\xi$$

This can be seen as a system in ambient space $R^{d-1,2}$ with coordinates $X^A = Y^A + V^A$.

If in addition

$$\frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X}\Phi = \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial P}\Phi = \frac{\partial}{\partial P} \cdot \frac{\partial}{\partial P}\Phi = 0$$

This precisely determines massless totally symmetric fields on AdS_d.

This explains why equations of the form $[F_i, F_j] = \mathcal{C}_{ij}^{k} F_k$ are actually relevant in HS theory. This is just because off-shell AdS fields are described by $sp(2)$-relations linearized around $F^0_{11} = \frac{1}{2}X^2, F^0_{12} = X \cdot P, F^0_{22} = \frac{1}{2}P^2$.

Note that $F_{22} = F^0_{22} + f_{22}$ is directly related to metric-like field. In some sense extended system contains both frame-like and metric-like formulations.

In its turn this is a familiar constrained system describing conformal scalar (singleton) on the boundary of AdS_d. Also known in the two-time approach of *I. Bars*.

Consistent factorization

The d-dimensional system is off-mass shell (at the linearized level fields are not treaceless).

Convenient to introduce $\bar{F}_{\alpha\beta} = F_{\alpha\beta} - T_{\alpha\beta}$ (basis in ideal $\mathfrak{h} \subset sp(2) \Subset osp(1|2)$)

$$[\bar{F}_{\alpha\beta}, \bar{F}_{\gamma\delta}]_\star = \epsilon_{\beta\gamma}\bar{F}_{\alpha\delta} + \dots\,, \qquad [\bar{F}_{\alpha\beta}, S_\gamma]_\star = 0\,, \qquad [\bar{F}_{\alpha\beta}, T_{\gamma\delta}]_\star = 0\,,$$

Eliminate the configurations proportional to $\bar{F}_{\alpha\beta}$. This is done by passing to equivalence classes of field configurations:

$$W \sim W + \lambda^i \star \bar{F}_i\,, \qquad T_a \sim T_a + \lambda^j_a \star \bar{F}_j\,,$$

where $T_a = \{S_\alpha, T_{\alpha\beta}, F_{\alpha\beta}\}$ and $\bar{F}_l = \{\bar{F}_{\alpha\beta}\}$.

The system is well-defined on equivalence classes:

$$dW + W \star W = u^l \star \bar{F}_l\,,$$

$$dT_a + [W, T_a]_\star = u^l_a \star \bar{F}_l\,,$$

$$[T_a, T_b]^{\pm}_\star - \mathcal{C}^c_{ab} T_c = u^l_{ab} \star \bar{F}_l\,,$$

The last equation and its symmetry is known in Hamiltonian constrained systems.

Interacting partially-massless (PM) fields

Partially massless fields: *Deser, Nepomechie (1983), Deser, Waldron (2001)*

$$\nabla^2\phi_{\mu_1\ldots\mu_s} + \ldots = 0\,, \qquad \delta\phi_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1}\ldots\nabla_{\mu_t}\xi_{\mu_{t+1}\ldots\mu_s)} + \ldots$$

Spin s depth t PM field. Usual massless fields $t=1$.

The factorization is determined by the ideal $\mathfrak{h}\subset\mathfrak{g}$ ($\mathfrak{h}$ is formed by $\bar{F}_{\alpha\beta}=F_{\alpha\beta}-T_{\alpha\beta}$). More generally, the system remains consistent if one factorize with respect to an ideal in $U(\mathfrak{g})$ generated by some $\bar{F}_L\in U(\mathfrak{g})$. In particular, the ideal generated by

$$\mathcal{F}_0 = -\frac{1}{2}\bar{F}_{\alpha\beta}\star\bar{F}^{\alpha\beta} - (\lambda^2-1)\,, \qquad \mathcal{F}_{\alpha_1\ldots\alpha_{2\ell}} = \bar{F}_{(\alpha_1\alpha_2}\star\ldots\star\bar{F}_{\alpha_{2\ell-1}\alpha_{2\ell})}$$

gives a theory of interacting partially-massless fields of all spins and depth $t=1,3,\ldots,2\ell-1$. Note that $t=1$ gives the usual massless fields.

This generalization is a bulk dual of nonunitary singleton $\Box^\ell\phi=0$ *Bekaert, M.G. (2013)*.

Conclusions

- Proposed a uniform representation of Vasiliev HS theories based on two algebras $\mathfrak{g}$ and $\mathcal{A}$.
- Expected to make AdS/CFT structures manifest. E.g. Vasiliev equations are merely components of BFV-BRST master equations for a singleton.
- In particular, studying the boundary behavior is straightforward in such approach cf. *Bekaert, M.G. 2013,2014* Nonliner off-shell system for Fradkin-Tseutlin fields on the boundary.
- May lead to understanding "generalized metric like" formulation of interacting HS theories through the new extended system in d dimensions. This is in turn related to parent formulations *Barnich, M.G. (also Alkalaev, Bekaert, M.G.)*
- Naturally has a simple Alexandrov-Kontsevich-Schwartz-Zaboronsky (AKSZ) representation based on $\Psi = W + c^a T_a + \ldots$ and $Q\Psi = \Psi \star \Psi - \frac{1}{2} c^a c^b \mathcal{C}^c_{ab} \frac{\partial}{\partial c^c}$.