On AdS_2 higher spin gravity

Konstantin Alkalaev

I.E.Tamm Theory Department, P.N.Lebedev Physical Institute

Based on arXiv:1311.5119
arXiv:1404.5330

Moscow 2014

Outline

- Lower dimensional gravity, the Jackiw-Teitelboim (JT) model.
- Higher spin extension of the JT model: (in)finite dim cases
- Metric-like versus frame-like formulation: a scalar/current duality
- Conclusions and outlooks

Lower dimensional gravity

The Einstein equations ($\Lambda = 0$)

$$G_{mn} \equiv R_{mn} - \frac{1}{2} g_{mn} R = 0 \,,$$

where R_{mn} is the Ricci, while R is the scalar: traces of the Riemann curvature $R_{mn,kl}$.

- In general dimensions $d \geq 4$: setting $G_{mn} = 0$ does not imply $R_{mn,kl} = 0$. The space needs not to be flat.
- For $d = 2, 3$: Weyl tensor vanishes identically, $C_{mn,kl} = 0$. The space is flat.
- For $d = 3$:

$$\epsilon^{m\alpha\beta}\, \epsilon_{n\gamma\rho}\, R_{\alpha\beta,}{}^{\gamma\rho} = G_n{}^m$$

The Chern-Simons formulation of the $3d$ gravity.

- For $d = 2$:

$$\epsilon^{\alpha\beta}\, \epsilon_{\gamma\rho}\, R_{\alpha\beta,}{}^{\gamma\rho} = R$$

so it follows that $G_{mn} \equiv 0$. What to do? The simplest diffeomorphism invariant equation is

$$R = const \,.$$

B. Barbashov, V. Nesterenko, A. Chervyakov'1979,
C. Teitelboim, '1983, R. Jackiw'1984

$$R + \Lambda = 0$$

The theory is not Lagrangian (# variables > # equations). Adding a scalar field one arrives at the particular dilaton gravity with the action for fields $g_{mn}(x)$ and $\phi(x)$

$$S_{JT}[\phi, g] = \int dx^2 \sqrt{-g}(R + \Lambda)\phi$$

The properties:

- no local PDoF
- AdS_2 and BH solutions (analogous to BTZ)
- an effective theory for $AdS_2 \times S^2$ near-horizon RN geometry
- In the conformal gauge, the JT equation is the Liouville equation+ residual diff inv.

BF action for $o(2,1) \approx sl(2,\mathbb{R})$ algebra of AdS_2 isometry $[T^A, T^B] = \epsilon_{ABC} T^C$:
(T. Fukuyama, K. Kamimura' 1985)

$$S_{JT}[\Psi, W] = \int_{\mathcal{M}^2} \Psi_A \mathcal{R}^A \,, \qquad \mathcal{R}^A = dW^A - \epsilon^{ABC} W_A \wedge W_C$$

The fields are 0-form $\Psi = \Psi_A T^A$ and 1-form $W = W_A T^A$ taking values in the adjoint of $sl(2,\mathbb{R})$. The field equations are

$$\mathcal{R}^A_{mn} = 0 \qquad D_m \Psi^A = 0$$

The original JT equation $\equiv$ $\mathcal{R}^{A=2}_{mn} = 0$.

Higher spin extension of the JT model

HS generalization of the JT theory is straightforward in the BF form.

The proposition

AdS_2 higher spin gravity $\quad \equiv \quad$ BF theory with $\mathcal{A}$-fields, where $\mathcal{A} = sl(N,\mathbb{R})$ Lie algebra.

A gauge algebra $\mathcal{A} = sl(N,\mathbb{R})$ in the higher spin basis:

- $N = 2$: in this case $\mathcal{A} = sl(2,\mathbb{R}) \approx o(2,1) = AdS_2$ global sym algebra
- $N \geq 3$:
$$T = T_{A_1} \oplus T_{A_1 A_2} \oplus \cdots \oplus T_{A_1 \dots A_{N-1}}$$
and there are $N-1$ generators in total. Here,
$$T_{A_1 \dots A_k} \; : \qquad T_{(A_1 \dots A_k)} \quad \text{and} \quad \eta^{MN} T_{MNA_3 \dots A_k} = 0$$
is a spin-k generator: the adjoint of $sl(2,\mathbb{R}) \subset sl(N,\mathbb{R})$. One can check (the principal embedding)
$$\# T = N^2 - 1 = \# \sum_{k=1}^{N-1} T_{A_1 \dots A_k} = \sum_{k=1}^{N-1} (2k+1)$$
- The $N = 3$ example: here dim $sl(3,\mathbb{R}) = 8$, there are 8 generators T^α, where $\alpha = 1, ..., 8$. In the higher spin basis
$$T^\alpha = T^A \oplus T^{(AB)}$$
Spin-2 generator T^A with $\# = 3$ and spin-3 generator T^{AB} with $\# = 5$.

BF fields & BF action

BF gauge fields:

- Zero-forms

$$\Psi(x) = \sum_{s=2}^{N-1} \Psi^{A_1\dots A_{s-1}}(x)\, T_{A_1\dots A_{s-1}}$$

- One-forms

$$W_m(x) = \sum_{s=2}^{N-1} W_m^{A_1\dots A_{s-1}}(x)\, T_{A_1\dots A_{s-1}}$$

Here, the expansion coefficients are the frame-like fields. Indices $A = 0, 1, 2$ and $m = 0, 1$.

BF higher spin action:

$$S[W, \Psi] = \sum_{s=2}^{N-1} \int_{\mathcal{M}^2} \Psi_{A_1\dots A_{s-1}} \mathcal{R}^{A_1\dots A_{s-1}}$$

Here,

$$\mathcal{R} = dW + [W, W], \qquad \delta W = d\xi + [\xi, W]\,, \qquad \delta\Psi = [\xi, \Psi]\,, \qquad \delta\mathcal{R} = [\xi, \mathcal{R}]$$

where a gauge parameter ξ is an $\mathcal{A}$-valued zero-form.

The BF equations of motion:

$$\mathcal{R}_{mn}^{A_1\dots A_{s-1}} = 0\,, \qquad D_m\Psi^{A_1\dots A_{s-1}} = 0\,, \qquad s = 2, ..., N$$

where $D = d + [W, \cdot]$ is the covariant derivative.

A few comments

- The JT gravity is embedded into BF HS gravity since $sl(2,\mathbb{R}) \subset sl(N,\mathbb{R})$

$$W_m = W_m^A T_A + W_m^{AB} T_{AB} + \ldots$$

  where all higher spin fields are set to zero.

- A natural background is AdS_2 spacetime.
- The BF HS theory is non-linear. One can linearize around the AdS background.
- Our main conclusion: BF theory with $\mathcal{A} = sl(N,\mathbb{R})$ gauge algebra is interpreted as dilaton higher spin gravity with $(N-1)$ *partially-massless* fields + dilaton fields.

Interpretation of the model

Consider the gauge sector of our model: fields $W_m^{A_1 \ldots A_{s-1}}$. Then, recall massless field formulations in d-dimensional AdS_d spacetime.

Massless HS fields: metric-like vs. frame-like (Fronsdal'1978, Vasiliev'2001)

Lorentz rank-s tensor fields — (i) totally symmetric, (ii) double traceless :

$$\phi_{m_1 \ldots m_s} \qquad \text{with the gauge symmetry} \qquad \delta\phi_{m_1 \ldots m_s} = \nabla_{(m_1}\xi_{m_2 \ldots m_s)}$$

These are metric-like (Fronsdal) higher spin fields. Consider now frame-like fields which are one-forms taking values in a particular $o(d-1,2)$ irrep

$$W_m^{A_1 \ldots A_{s-1}, B_1 \ldots B_{s-1}} \qquad \text{with the gauge symmetry} \qquad \delta W_m = D_m \xi$$

- For $d=2$: all $W_m^{A_1 \ldots A_{s-1}, B_1 \ldots B_{s-1}} \equiv 0$ except for $s=2$ (gravity) case.
- For $d=2$ and $s=2$: the Hodge duality $W_m^{A_1, B_1} = \epsilon^{A_1 B_1 C} W_{m\,C}$.

Partially-massless HS fields (Deser, Nepomechie, Waldron, Zinoviev, Vasiliev, Skvortsov, 1983 - 2006)

Field W_m^A belongs to

$$W_m^{A_1 \ldots A_{s-1}}\,, \qquad \text{where} \qquad s = 2, 3, \ldots$$

In d dimensions these forms are partially massless gauge fields of the maximal depth. Their metric-like form is given by $\phi_{m_1 \ldots m_s}$ with $\delta\phi_{m_1 \ldots m_s} = \nabla_{(m_1} \cdots \nabla_{m_s)}\xi + \ldots$.

An infinite-dimensional extension

The gauge algebra $sl(N,\mathbb{R})$ can be infinitely extended:

An infinite-dimensional HS algebra: Feigin'1988, Vasiliev' 1989

Gauge algebra $\mathcal{A} = \mathrm{hs}[\nu]$, where $\nu = m(m+1)$ for $m \in \mathbb{R}$

- $\#$ fields $= \infty$ for a generic m. There are ∞ many HS generators

$$\bigoplus_{s=2}^{\infty} T_{A_1 \ldots A_{s-1}}$$

  A field of each spin s enters in a single copy.

- $\#$ fields $< \infty$ for a $m = 0, 1, 2, \ldots$. In this case $\mathcal{A}$ is not simple:

$$\mathrm{hs}[\nu]/\mathcal{I} = sl(m+2,\mathbb{R})$$

  There are $m+2$ spin-s fields, $s = 2, ..., m+2$.

- The action reads (Howe dual $sp(2) - o(2,1)$ realization of $\mathrm{hs}[\nu]$: Alkalaev'2014)

$$S_\nu[\Psi, W] = \int_{\mathcal{M}^2} \mathrm{Tr}\Big[\Delta_\nu \, \Psi \, \mathcal{R}(W)\Big] \,, \quad \text{where} \quad \Delta_\nu - \text{some projecting operator.}$$

Linearized dynamics

Fluctuations

$$W = W_0 + \Omega \,, \qquad \Psi = \Psi_0 + \Phi$$

where (W_0, Ψ_0) is a background. We choose W_0 = AdS spacetime, $\Psi_0 = 0$.

The linearized equations of motion for spin-s decoupled subsystems, $s = 2, ..., N-1$:

$$D_0 \Phi^{A_1 \dots A_{s-1}} = 0 \qquad \text{and} \qquad R^{A_1 \dots A_{s-1}} \equiv D_0 \Omega^{A_1 \dots A_{s-1}} = 0$$

where $D_0 = d + W_0$ is the background covariant derivative, $D_0 D_0 = 0$. The gauge symmetry transformations read

$$\delta \Omega^{A_1 \dots A_{s-1}} = D_0 \xi^{A_1 \dots A_{s-1}} \qquad \text{and} \qquad \delta \Phi^{A_1 \dots A_{s-1}} = 0$$

Lorentz decomposition

Spin-2 case: the zweibein and the spin connection

$$\Omega^A_m \quad \to \quad e^a_m \oplus \omega_m \qquad\qquad A = 0,1,2 \,, \quad a, ..., m... = 0,1$$

Spin-s case: $o(2,1)$ fields decompose into $o(1,1) \subset o(2,1)$ components

$$\Omega_m^{A_1 \dots A_{s-1}} = \omega_m \oplus \omega_m^{a_1} \oplus \omega_m^{a_1 a_2} \oplus ... \oplus \omega_m^{a_1 \dots a_{s-1}}$$

$$R_{mn}^{A_1 \dots A_{s-1}} = R_{mn} \oplus R_{mn}^{a_1} \oplus R_{mn}^{a_1 a_2} \oplus ... \oplus R_{mn}^{a_1 \dots a_{s-1}}$$

Two different ways to reduce BF system

Let us consider the gauge sector of the model. The field equations in the Lorentz basis are

$$R_{mn}^{a_1 \dots a_k}(\omega) = 0\;, \qquad k = 0, 1, ..., s-1\;.$$

Low spin examples:
(s=1) $R_{mn} \equiv F_{mn} = 0$ is Maxwell BF theory.
(s=2) $R_{mn} = 0$ and $R^a_{mn} = 0$ is the Jackiw-Teitelboim theory.

A triplet form of the field space of BF system

Field space = (dynamical fields) $\oplus$ (auxiliary fields) $\oplus$ (Stueckelberg fields)

First reduction: dynamical fields ϕ and $\phi_{a_1\dots a_s}$

Using the higher spin gauge $\phi_{a_1\dots a_s} = 0$ one arrives at the KG equation

$$\nabla^2\phi - m_s^2\phi = 0\;, \qquad \text{where} \qquad m_s^2 = s(s-1)\Lambda\;, \quad s \geq 2\;,$$

plus leftover gauge symmetry satisfying generalized Killing eqs.

Second reduction: dynamical fields φ and $\varphi_{a_1\dots a_s}$

Using the scalar gauge $\varphi = 0$ one arrives at the conservation condition

$$\nabla^n\varphi_{na_1\dots a_{s-1}} = 0\;,$$

plus leftover gauge symmetry expressed as particular "improvements".

A few comments

- The original (linearized) BF higher spin theory gives rise to two metric-like theories related by a duality transformation: *scalar/current duality*.
- BF equations (and action) are "parent" for two dual metric-like formulations (in the spirit of Fradkin and Tseytlin'1986).
- This is similar to WZW model: $g(x)$ satisfies the second-order eq $\partial^m(g^{-1}\partial_m g) = 0$. On the other hand, introducing a current $J_m = g^{-1}\partial_m g$ one obtains a conservation condition $\partial^m J_m = 0$.
- The theory has no local PDoF. It is obvious for BF formulation. Within the metric-like formulations there are gauge symmetries that eliminate all local degrees of freedom.

Conclusions & outlooks

Done:

- Higher spin gravity in AdS_2 spacetime formulated as BF-type theory with fields taking values either in finite-dim or infinite-dim higher spin algebra.
- Vector description of $2d$ HS algebra $hs[\lambda]$ using Howe dual algebras $o(2,1) - sp(2)$.
- The linearized metric-like dynamics: dual scalar/current descriptions. It follows form the $\sigma_{\pm}$ cohomology problem.

To be done:

- Black hole type solutions to AdS_2 higher spin gravity which generalize known black hole solutions to the Jackiw - Teitelboim gravity. Analogous to BTZ black holes.
- The AdS_2/CFT_1 for a one-parametric HS algebra $\mathrm{hs}[\nu]$: an explicit description of the corresponding classical mechanics.